Some Results in Harmonic Analysis Related to Pointwise Convergence and Maximal Operators

by

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Abstract

Pointwise convergence problems are of fundamental importance in harmonic analysis and studying the boundedness of associated maximal operators is the natural viewpoint from which to consider them. The first part of this two-part thesis pertains to Lennart Carleson's landmark theorem of 1966 establishing almost everywhere convergence of Fourier series for functions in $L^2(\mathbb{T})$. Here, partial progress is made towards adapting the time-frequency analytic proof of Carleson's result by Michael Lacey and Christoph Thiele to bound an almost periodic analogue of Carleson's maximal operator for functions in the Besicovitch space B^2. A model operator of the type of Lacey and Thiele is formed and shown to relate to Carleson's operator in a natural way and be susceptible to a similar kind of analysis.

In the second part of this thesis, recent work of Per Sjölin and Fernando Soria is improved, with precise boundedness properties determined for the Schrödinger maximal operator with complex-valued time as a special case of more general estimates for a family of maximal operators associated to dispersive partial differential equations. Boundedness properties of other maximal operators naturally related to the Schrödinger maximal operator are also established using similar techniques.

L'analyse mathématique est aussi étendue que la nature elle-même; elle définit tous les rapports sensibles, mesure les temps, les espaces, les forces, les températures. [...] *Elle rapproche les phénomènes les plus divers, et découvre les analogies secrètes qui les unissent. Si la matière nous échappe comme celle de l'air et de la lumière par son extrême ténuité, si les corps sont placés loin de nous, dans l'immensité de l'espace, si l'homme veut connaître le spectacle des cieux pour des époques successives que sépare un grand nombre de siècles, si les actions de la gravité et de la chaleur s'exercent dans l'intérieur du globe solide à des profondeurs qui seront toujours inaccessibles, l'analyse mathématique peut encore saisir les lois de ces phénomènes.*

Joseph Fourier, *Théorie Analytique de la Chaleur* (1822), pp. xiv–xv.

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Contents

Note: blank pages from the print version of this thesis have been omitted in this electronic version. To ensure that the two versions have consistent page numbering, page numbers here are not strictly sequential.

Remarks on Notation

The notation $f \lesssim g$ will be taken to mean that $f \leqslant Cg$ for some fixed positive constant C. Analogously, $f \approx g$ will mean $f = Cg$ for some fixed $C > 0$. In both cases, the constant C is generally assumed to be independent of any functions, variables and parameters unless otherwise indicated by subscripts. The notation $f \sim g$ will mean that $f \lesssim g$ and $g \lesssim f$ both hold.

The symbol $\mathscr{S}(\mathbb{R})$ is used to represent the Schwartz space of rapidly decaying functions, namely the space of functions $f \in C^\infty(\mathbb{R})$ satisfying the property that for any n, $m \in \mathbb{N}_0$, the quantity $\sup_{x \in \mathbb{R}} |x^n f^{(m)}(x)|$ is finite. For a suitable measure space (X, Σ, μ), the symbol $\mathscr{M}(X, \mu)$ will be used to denote the set of all μ-measurable functions mapping from X into $\mathbb{C}$, defined up to μ-null sets. Here, $\mathbb{C}$ can generally be replaced with $\mathbb{R}$ without issue, but it is important that functions contained in $\mathscr{M}(X, \mu)$ are not permitted to be infinite on sets of positive measure. When X is a Euclidean space, the symbol $\mathscr{M}(X)$ will be used to mean $\mathscr{M}(X, \mu)$ where μ is Lebesgue measure. For $s \geqslant 0$, the symbol $H^s(\mathbb{R})$ will be used to denote the inhomogeneous $L^2(\mathbb{R})$ Sobolev space of index s, namely the space of $L^2(\mathbb{R})$ functions such that the norm $\|f\|_{H^s(\mathbb{R})} := \|((1 + |\cdot|^2)^{\frac{s}{2}} \widehat{f})^\vee\|_{L^2(\mathbb{R})}$ is finite. The homogeneous $L^2(\mathbb{R})$ Sobolev norm will also sometimes be applied to functions in $H^s(\mathbb{R})$, that is to say the norm $\|f\|_{\dot{H}^s(\mathbb{R})} := \|(|\cdot|^s \widehat{f})^\vee\|_{L^2(\mathbb{R})}$, although the homogeneous $L^2(\mathbb{R})$ Sobolev space, $\dot{H}^s(\mathbb{R})$, will not be considered directly here. To pass between these two norms, the elementary fact that $\|f\|_{H^s(\mathbb{R})} \sim \|f\|_{L^2(\mathbb{R})} + \|f\|_{\dot{H}^s(\mathbb{R})}$ will be used.

The circle group is denoted by $\mathbb{T}$ with functions on $\mathbb{T}$ being identifiable with 1-periodic functions on $\mathbb{R}$. It will be implicitly equipped with the normalised Lebesgue measure throughout. It should be noted that throughout the first part of this thesis, the symbol $\mathbb{T}$ will also often be used to represent an arbitrary tree in the time-frequency plane. There should be no ambiguity in any case over which purpose the symbol is serving.

The notation $\widehat{f}$ will be used interchangeably for the Fourier transform of a function f, defined as $\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi}\, dx$ (here, d will usually be 1); the function on $\mathbb{Z}$ defining the Fourier coefficients of a function f on $\mathbb{T}$, defined as $\widehat{f}(n) \coloneqq \int_{\mathbb{T}} f(x)e^{-2\pi inx}\,dx$; and, in the first part of this thesis, the function on $\mathbb{R}$ corresponding to the Fourier coefficients of an almost periodic function f on $\mathbb{R}$, defined as $\widehat{f}(\lambda) \coloneqq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)e^{-2\pi i\lambda x}\,dx$. The meaning in each instance should be clear from the context.

With respect to the first part of this thesis, it is remarked that the notation used in the literature in both the study of almost periodic functions and that of time-frequency analysis has not been standardised. Herein, the choice of notation ascribes to personal preference and instinct, in the former case conforming to the style introduced in the author's previous thesis[11] where applicable. The nature of this part of the thesis is such that it requires a significant amount of non-standard notation, some of which is new. All such notation is defined on its first use, but for reference purposes, a table of notation is provided on page 137.

Introduction

Since the birth of the subject, a fundamental question in harmonic analysis has always been "when does the Fourier series associated with a given function converge pointwise?" This question has turned out to be a deep one; satisfactory answers to questions of this form have often taken decades to develop and even now, over two hundred years since the start of Fourier's pioneering work on the conduction of heat, there remain open problems at the heart of harmonic analysis of this type.

The first part of this two-part thesis is based on one of the landmark results on pointwise convergence of Fourier series, first proved in 1966 by Lennart Carleson:

Carleson's Theorem (1966) *The Fourier series of any function in $L^2(\mathbb{T})$ converges almost everywhere.*[44]

Carleson's proof was refined by Charles Fefferman in 1973[57] and then again by Michael Lacey and Christoph Thiele in 2000[92]. The proof of Lacey and Thiele makes use of time-frequency analytic techniques developed from the ideas of Carleson and Fefferman and proceeds by modelling the $L^2(\mathbb{R})$ Carleson maximal operator,

$$\mathscr{C}_{\mathbb{R}} f \coloneqq \sup_{N \in \mathbb{R}} \left| \int_{\xi \leqslant N} \widehat{f}(\xi) e^{2\pi i \xi \cdot} \, d\xi \right|,$$

by an operator that is composed of pieces that are simultaneously well-localised in time and frequency space and subsequently proving its boundedness as an operator mapping from $L^2(\mathbb{R})$ into $L^{2,\infty}(\mathbb{R})$.

In the present thesis, Lacey and Thiele's methods will be adapted to the almost periodic Carleson maximal operator, defined for an almost periodic function f as

$$\mathscr{C} f = \sup_{\xi \in \mathbb{R}} \left| \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n < \xi}} \widehat{f}(\lambda_n) e^{2\pi i \lambda_n \cdot} \right|.$$

Here, this operator will be considered to be acting on functions from the Besicovitch space, B^2 and partial progress will be made towards adapting the method of Lacey and Thiele to establish its weak boundedness. A time-frequency model for this operator of the form of Lacey and Thiele will be proposed and shown to both model this operator in a natural way and be susceptible to a type of analysis analogous to that from [92]; this will involve establishing almost periodic versions of the mass, energy and tree lemmata of Lacey and Thiele.

Part I of this thesis is divided into four chapters. The first chapter will discuss the background behind this work. In particular, Section 1.1 will examine some aspects of the historical development of problems related to pointwise convergence of Fourier series. Sections 1.2 and 1.3 will provide the mathematical framework necessary for the remainder of the first part of this thesis, the former presenting an outline of Lacey and Thiele's proof of boundedness of the $L^2(\mathbb{R})$ Carleson operator and the latter providing a concise introduction to the theory of almost periodic functions with references to further reading on the subject. Section 1.3 will also discuss some more specialised results pertaining to almost periodic functions that will be relevant to the subsequent chapters of Part I.

These remaining chapters will provide the details of the proposed scheme for proving weak B^2 boundedness of the almost periodic Carleson operator. Chapter 2 introduces the problem and describes the process of localisation of functions in the almost periodic time-frequency plane. Some estimates on the almost periodic localising function are also established. In Chapter 3, a time-frequency decomposed operator that models summation of almost periodic Fourier series is formulated. Its boundedness and symmetry properties are explored and, through a process of averaging, it is shown that bounding a maximal version of this operator in a particular way is equivalent to bounding the Carleson operator. Analogues of the mass, energy and tree lemmata from Lacey and Thiele's work are stated and a boundedness property of the maximal model operator is shown to follow from these lemmata by grouping the localised parts of the model operator into suitable collections. Given natural slightly stronger versions of the energy and tree lemmata, this boundedness property could be improved to that required to conclude boundedness of the Carleson operator. Chapter 4 is devoted to the proofs of the mass, energy and tree lemmata.

Whilst it is not entirely clear what convergence property might follow from weak boundedness of the Carleson operator in the Besicovitch spaces of almost periodic functions, in the regular L^p spaces there is an intimate relationship between boundedness of maximal operators and pointwise convergence that is not confined to the problem of convergence of Fourier series. For a measurable space, (X,μ), let T_j be a sequence of operators acting on functions in $L^p(X,\mu)$ such that $\lim_{j\to\infty} T_j f(x) = f(x)$ for μ-almost every x whenever f is a member of some dense subspace of $L^p(X,\mu)$. It is a standard result that if the maximal operator $T^* f(x) \coloneqq \sup_j |T_j f(x)|$ satisfies the bound

$$\|T^* f\|_{L^{q,\infty}(X,\mu)} \lesssim \|f\|_{L^p(X,\mu)}$$

for any particular $q \in [1,\infty]$, then $\lim_{j\to\infty} T_j f(x) = f(x)$ for μ-almost every x for *all* $f \in L^p(X,\mu)$ (see, for example, [55, Thm. 2.2, p. 27]). In many convergence problems in harmonic analysis, convergence in a dense subspace of L^p is straightforward to establish (for example, the Fourier series of any trigonometric polynomial clearly converges everywhere), so this principle shows that convergence problems can be solved by considering boundedness problems for maximal operators. Furthermore, the correspondence between these two types of problems is deeper than this simple technical result suggests and in many situations, as will be seen, pointwise convergence does not just follow from boundedness of a maximal operator, but is actually equivalent to it.

The second part of this thesis considers questions of this form arising from the free Schrödinger equation,

$$i\partial_t u(t,x) = \Delta_x u(t,x).$$

A classical question here is to consider the associated initial value problem and determine the regularity required of the initial data, $u(0,x) = f(x)$, for the solution, $u(t,x)$, to converge pointwise almost everywhere to $f(x)$ as $t \to 0$. Here also Carleson was one of the pioneers, establishing in 1980 by means of a bound on a maximal operator that in one spatial dimension, it is sufficient that f be in the L^2 Sobolev space $H^{\frac{1}{4}}(\mathbb{R})$.[46] This condition was also shown to be necessary by Björn Dahlberg and Carlos Kenig two years later[51] and whilst this conclusion resolved the one dimension problem completely, the higher-dimensional question is still not fully answered.

The problems considered in Part II are inspired by work of Per Sjölin and Fernando Soria from [132] and [133] on boundedness of the Schrödinger maximal operator in one spatial dimension with complex-valued time. For time variable $t + it^\gamma$ for some fixed positive γ and $t \in (0,1)$, this maximal operator is of a form that is somewhere between the regular Schrödinger maximal operator and the maximal operator corresponding to the solution operator for the heat equation. The latter operator is controlled by the Hardy–Littlewood maximal operator and is thus, for example, bounded as a map from $L^2(\mathbb{R})$ to $L^2([-1,1])$ whilst the Schrödinger maximal operator is bounded as a map from $H^s(\mathbb{R})$ to $L^2([-1,1])$ if and only if $s \geqslant \frac{1}{4}$. It is thus natural to ask for what $s(\gamma)$ the Schrödinger maximal operator with time $t + it^\gamma$ is bounded as a map from $H^{s(\gamma)}(\mathbb{R})$ to $L^2([-1,1])$. Here, this question is answered by solving the same problem for the operators

$$P^*_{a,\gamma} f(x) := \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{-t^\gamma|\xi|^a} e^{ix\xi} \, d\xi \right|$$

with parameter $a > 1$. These operators are the complex time analogues of the operators

$$S^*_a f(x) := \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{ix\xi} \, d\xi \right|$$

which in turn are the maximal operators corresponding to the solution operators for the dispersive partial differential equations,

$$i\partial_t u(t,x) + (-\Delta)^{\frac{a}{2}} u(t,x) = 0.$$

In the case of $a = 2$, $P^*_{a,\gamma}$ corresponds to the Schrödinger maximal operator with complex-valued time considered by Sjölin and Soria; the boundedness properties established in this thesis originate from a recent paper by the author[9] and complete as a special case the partial resolution by Sjölin and Soria of their problem.

The techniques involved in the proofs of these results are also adapted to establish boundedness properties of other natural generalisations of the Schrödinger maximal operator, namely the operators

$$T^*_a f(x) \coloneqq \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} t^{-\frac{1}{a}} e^{\frac{i|y|^a}{t}} f(x-y)\, dy \right|.$$

These operators also correspond to the Schrödinger maximal operator in the case of $a = 2$ (albeit with real-valued time) and were considered previously by Luis Vega in his thesis from 1988[143]. It turns out that these operators can be represented in a form quite similar to the operators S^*_a and they are hence susceptible to similar methods of analysis.

Amongst these methods used to analyse the operators in Part II is a general scheme which is developed to extend the scope of the aforementioned work of Dahlberg and Kenig, providing counterexamples to boundedness of more general operators. This method uses techniques in non-linear optimisation to reduce showing failure of boundedness of an operator to solving a system of polynomial equations.

Part II begins with Section 5.1 on the background behind the pointwise convergence problem for the Schrödinger equation. Section 5.2 discusses in some detail the aforementioned equivalences between bounds for maximal operators and pointwise convergence results, providing information on the maximal principles of Stein, Sawyer and Nikishin.

Chapter 6 introduces and resolves the problem of boundedness of the Schrödinger-like maximal operators with complex-valued time, $P^*_{a,\gamma}$. The proof of the main theorem of the chapter is divided into two main sections: Section 6.2 deals with cases where boundedness can be established whilst Section 6.3 provides counterexamples that show where boundedness fails.

Chapter 7 considers boundedness of the operators T^*_a. Section 7.2 provides the calculations necessary to reduce boundedness of the operators T^*_a to operators that appear more like the operators S^*_a; the subsequent proof of the boundedness results for these operators is divided into two sections, as in Chapter 6, with the positive results proved in Section 7.3 and the negative results proved in Section 7.4. Section 7.5 discusses the aforementioned scheme used to generate counterexamples in both Chapter 6 and Chapter 7, providing a framework suitable for further generalisation. Section 7.6 generalises the work of Chapter 7 to operators with complex-valued time, drawing directly on the results and techniques of Chapter 6.

Part I

Carleson's Theorem for Almost Periodic Fourier Series

Chapter 1

Historical and Mathematical Background

1.1 Convergence of Fourier Series

Asking under what circumstances and in what manner the Fourier series of a function converges is one of the most fundamental questions in Fourier analysis and despite the longevity of the problem, there are still some pertinent open problems in the area. Whilst the question is entirely natural in today's mathematical culture, Joseph Fourier's own lack of analytical rigour in an early essay written for a competition set by the *Institut de France* in 1811, *Théorie du Mouvement de la Chaleur dans les Corps Solides*, led to the committee examining the paper, consisting of Joseph-Louis Lagrange, Pierre-Simon Laplace, Étienne-Louis Malus, René Just Haüy and Adrien-Marie Legendre, making the following comment in their report after deciding to award him the prize:

> *Cette pièce renferme les véritables équations différentielles de la transmission de la chaleur, soit à l'intérieur des corps, soit à leur surface; et la nouveauté de l'objet, jointe à son importance, a déterminé la classe à couronner cet ouvrage, en observant cependant que la manière dont l'auteur parvient à ses équations, n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du côté de la rigueur.*[1, p. 374]
>
> [*This work contains the true differential equations of heat transmission, either inside or from the surface of a body; and the novelty of the subject, coupled with its*

> *importance, caused [the examining group] to award this work the prize, observing however that the way in which the author arrives at his equations is not exempt from difficulties and that his analysis for integrating them leaves something to be desired, both relative to generality and in terms of rigour.*]

Of course, Fourier's lack of rigour was not surprising, given that the essay was written at a time when the rigorous foundations of calculus were still being set, and whilst the Institute decided not to publish the essay, it did ultimately lead to his *magnum opus*, *Théorie Analytique de la Chaleur*[61], published in 1822, the book now generally regarded as the birthplace of the Fourier series. Fourier's rigour, even here, would not stand up to modern scrutiny*, but by this point it was clear that he had an understanding of convergence fairly close to the modern accepted definition, stating,

> *Il est nécessaire que les valeurs auxquelles on parvient, en augmentant continuellement le nombre de termes, s'approchent de plus en plus d'une limite fixe, et ne s'en écartent que d'une quantité qui peut devenir moindre que toute grandeur donnée: cette limite est la valeur de la série.*[61, p. 247]
>
> [*It is necessary that the values at which we arrive, by continually increasing the number of terms, approach more and more a fixed limit, and only differ from it by a quantity that can become less than any given magnitude: this limit is the value of the series.*]

The notion of convergence that Fourier is speaking of is that of pointwise convergence. Stated in modern language, the question at hand is the following: given a complex-valued function f defined on $\mathbb{T}$ (identified with a periodic function on $\mathbb{R}$), for what values of $x \in \mathbb{T}$ is it true that

$$\lim_{N\to\infty} \sum_{|n|\leqslant N} \widehat{f}(n) e^{2\pi i n x} = f(x)?$$

One of the very early attempts to consider this issue for a reasonably general class of functions, rather than the very specific cases that Fourier attempts to deal with individually in [61], was in a paper by Augustin-Louis Cauchy published in 1823[47]. In it, he claims convergence of Fourier series for functions satisfying certain strong technical assumptions. However, in his own paper published in 1829[53], Johann Peter Gustav Lejeune Dirichlet reflects

*For example, even in 1906, Henri Lebesgue objected to Fourier's methods in [94, pp. 26–30].

on the fact that, by Cauchy's own admission, the results in [47] only apply to a limited class of functions, and then further comments that "une examen attentif du Mémoire cité m'a porté à croire que la démonstration qui y est exposée n'est pas même suffisante pour les cas auxquels l'auteur la croit applicable."[53, p. 158] [A careful examination of the dissertation cited has led me to believe that the proof given there is not even sufficient for the cases that the author believes apply.] He proceeds to establish the following theorem:

Theorem (Dirichlet, 1829) *The Fourier series of a piecewise continuous function with finitely many minima and maxima converges at every x to* $\dfrac{f(x-)+f(x+)}{2}$.*

Another paper by Dirichlet was published in 1837[54] in which this result was generalised slightly; some further generalisation by Camille Jordan in 1881[73] led to the now well-known Dirichlet–Jordan Theorem establishing pointwise convergence of Fourier series for functions of bounded variation. Another pointwise convergence result was proved by Rudolf Lipschitz in a paper published in 1864[100], in which it was shown that Fourier series for Hölder continuous functions converge pointwise. This work was generalised by Ulisse Dini in 1880[52], leading to the also now well-known Lipschitz–Dini convergence criterion that the Fourier series of a function f converges at a point x if the function $\dfrac{f(x+t)-f(x)}{t}$ is integrable in t near 0. However, in terms of understanding pointwise convergence of Fourier series for continuous functions – a problem that was to remain elusive until well into the twentieth century – perhaps more influential than these positive results was that in 1873, Paul du Bois-Reymond established in [22] that the Fourier series of a continuous function could diverge at a point. A simple example of a continuous function satisfying this property was also given by Lipót Fejér in 1911[58].

With the birth of measure theory at the turn of the twentieth century (and with examples of continuous functions with Fourier series that diverge at more than isolated points not being forthcoming), it became natural to ask under what circumstances Fourier series converged *almost* everywhere. In 1913, a note by Nikolai Luzin[102] established necessary and sufficient

*The aforementioned 1823 paper[47] was not the only work of Cauchy that was shown to be erroneous by Dirichlet in [53]. This theorem directly contradicts Cauchy's claim in his pivotal *Cours d'Analyse*, that the sum of a series of continuous functions is necessarily continuous[48, Thm. 1, p. 120].

conditions for almost everywhere convergence of the Fourier series of a square-integrable (that is to say L^2) function, f, namely that if g is the conjugate function of f, then

$$\frac{1}{\pi}\int_0^{\pi} \frac{g(x+\alpha)-g(x-\alpha)}{2\tan(\frac{\alpha}{2})}\,d\alpha = f(x) \text{ almost everywhere}$$

and

$$\lim_{n\to\infty} \frac{1}{\pi}\int_0^{\pi} \frac{g(x+\alpha)-g(x-\alpha)}{\alpha}\cos(n\alpha)\,d\alpha = 0 \text{ almost everywhere,}$$

where the integrals are taken in the principal-value sense.

After stating these conditions, he goes on to observe that the first condition is in fact satisfied for all $f \in L^2$ and states that it is "*infiniment probable*"[102, p. 1657] [infinitely probable] that the same is true for the second condition. The validity of this remarkably confident conjecture would, as Luzin comments, imply the following:

Luzin's Conjecture (1913) *For any $f \in L^2(\mathbb{T})$,*

$$\lim_{N\to\infty} \sum_{|n|\leqslant N} \widehat{f}(n)e^{2\pi i n x} = f(x)$$

for almost every $x \in \mathbb{T}$. In particular, the Fourier series of a continuous function converges almost everywhere.

Luzin went on to produce his PhD thesis, *Интеграл и Тригонометрический Ряд* [Integral and Trigonometric Series], in 1915 (reprinted in full in 1951 in [103]). In it he discusses his earlier conjecture and, at first, maintains his optimism, stating that "все результаты, полученные до сих пор в теории тригонометрических рядов, подтверждают вероятность этой гипотетической теоремы."[103, p. 219] [All results established so far in the theory of trigonometric series confirm probability of this hypothetical theorem.] However, he later seems to retract some of his confidence. Indeed, at the end of the thesis, there is a list of fifty two open questions, mostly pertaining to convergence of Fourier series. The first of these questions is the following, asking about a strongly negative conjecture (here, S_n is the n^{th} partial sum of the Fourier series of f):

Пусть $f(x)$ такова, что

$$\int_0^{2\pi} f^2(x)\,dx < +\infty$$

где $\{a_0, a_n, b_n\}$ определены по Фурье.

Может ли случиться, что для всякого x,

$$\overline{\lim_{n\to\infty}}\, S_n(x) = +\infty$$

и

$$\underline{\lim_{n\to\infty}}\, S_n(x) = -\infty \text{ ?}$$[103, p. 366]

[*Let $f(x)$ be such that*

$$\int_0^{2\pi} f^2(x)\,dx < +\infty$$

where $\{a_0, a_n, b_n\}$ are the Fourier coefficients of f.

Can it happen that for every x,

$$\overline{\lim_{n\to\infty}}\, S_n(x) = +\infty$$

and

$$\underline{\lim_{n\to\infty}}\, S_n(x) = -\infty \text{ ?}]$$

Whatever Luzin's instinct on his original conjecture ultimately was, an answer did not come easily and two significant *divergence* results were proved in the years that followed, casting some doubt on the veracity of Luzin's original "*infiniment probable*" conjecture. In 1923, at the age of only 20, Andrey Kolmogorov published an example of a function in L^1 that had a Fourier series that diverged almost everywhere[83] and in 1926, he improved this result to divergence everywhere[82].

In spite of this, there were some convergence results established prior to the final resolution of Luzin's conjecture. In 1924, between the publication of his two divergence results, Kolmogorov managed to establish almost everywhere convergence of Fourier series of L^2 functions when the partial sums were taken in a lacunary sense[84] (that is to say that there is a fixed constant $\lambda > 1$ such that the infimum of the ratio of the number of terms in the n^{th}

partial sum to that in the $(n-1)^{\text{th}}$ exceeds λ).* Additionally, in 1926, Kolmogorov and Gleb Aleksandrovich Seliverstov jointly[85], as well as Abraham Plessner independently[118], established that the Fourier series of a function in $L^2(\mathbb{T})$ with real Fourier expansion coefficients (a_n), (b_n) converges almost everywhere if

$$\sum_{n=1}^{\infty} \log(n)(a_n^2 + b_n^2) < \infty.$$

The Kolmogorov–Seliverstov–Plessner result remained unimproved for the next forty years and by 1959, Antoni Zygmund, at least, seems to have been pessimistic about the possibility of a positive resolution to Luzin's original conjecture. He discusses it briefly in the second volume of *Trigonometric Series*[149, pp. 164–166], referring to it as "the still unsolved problem of the existence of an $f \in L^2$ with $\mathbf{S}[f]$ divergent almost everywhere"[149, p. 165] (where $\mathbf{S}[f]$ is used by Zygmund to denote the limiting value of the partial sums of the Fourier series of f). He concludes with a condition that would prove the existence of such a function. It also appears that Elias Stein may have been of a similar viewpoint, referring to the problem in a similar way in a paper published in 1961[135, p. 142].

Their shared pessimism about the veracity of Luzin's conjecture aside, these works of Zygmund and Stein did share an important role in contributing to a deeper understanding of the problem. In the course of his discussion of the conjecture, Zygmund presented a result of Alberto Calderón[149, Thm. XIII.1.22, p. 165] showing that a positive resolution would imply weak L^2 boundedness of a maximal operator.† Stein's contribution was to generalise this result to a wider setting. Combined with an early result of Stefan Banach[13], this generalisation can be stated as the following theorem‡, which is a form of what is now widely known as the Stein Weak-Type Maximal Principle, establishing an equivalence of bounds on maximal operators and pointwise convergence results:

*It is this result that the author's previous thesis[11] (and subsequent paper, [10]) is based on.

†Calderón does not seem to have published this result independently. Zygmund credits him for the result in the notes provided at the end of [149].

‡Specifically, the theorem as stated combines Corollary 1 from [135] with Theorem III from [13] and the straightforward observation that weak-type boundedness of the maximal operator trivially implies its finiteness.

Theorem (Calderón, Stein) *Let $p \in [1,2]$ and suppose that (T_n) is a sequence of bounded linear operators on $L^p(\mathbb{T})$ that commute with translations and are such that $\lim_{n\to\infty} T_n f$ exists almost everywhere for all f in some dense subspace of $L^p(\mathbb{T})$.*

Then $\lim_{n\to\infty} T_n f$ exists almost everywhere for all $f \in L^p(\mathbb{T})$ if and only if the maximal operator $\sup_n |T_n f|$ is of weak-type (p,p).

Results of this type will be of greater importance in Part II and will be discussed in more detail in Section 5.2.

In 1965, Jean-Pierre Kahane and Yitzhak Katznelson submitted a pair of papers to *Studia Mathematica* which provided substantial insight into the problem of pointwise convergence of Fourier series for both continuous functions and L^p functions (for $p \in (1,\infty)$; the $p = 1$ problem was resolved by Kolmogorov's results). The first paper[78], authored solely by Katznelson, defines $E \subseteq \mathbb{T}$ to be a *set of divergence* for a class of functions, B, if there exists a function $f \in B$ whose Fourier series diverges at every point of E. The following proposition is then established:

Proposition (Katznelson) *Let $B \in \{L^p(\mathbb{T}) : p \in (1,\infty)\} \cup \{C(\mathbb{T})\}$. Then if there exists a set of divergence for B of positive measure, there exists one of full measure. It follows that either all Fourier series of functions in B converge almost everywhere, or there exists a function in B whose Fourier series diverges almost everywhere.**

Katznelson further establishes that, under the additional hypothesis on B that every set of zero measure is a set of divergence for B, the result can be strengthened to divergence everywhere. After verifying this hypothesis for L^p, $p \in (1,\infty)$, he establishes the paper's headline result:

Theorem (Katznelson, 1966) *For each $p \in (1,\infty)$, either the Fourier series of every function in $L^p(\mathbb{T})$ converges almost everywhere, or there exists a function in $L^p(\mathbb{T})$ whose Fourier series diverges everywhere.*

*Katznelson's original proposition is more general than the statement given here: he proves the result for any function space B satisfying certain hypotheses.

The second paper[76], attributed to Kahane and Katznelson jointly, verifies the hypothesis for the space of continuous functions, $C(\mathbb{T})$, giving its headline result as the following:

Theorem (Kahane and Katznelson, 1966) *Either the Fourier series of every continuous function converges almost everywhere, or there exists a continuous function whose Fourier series diverges everywhere.*

As interesting – and potentially useful – as these dichotomies were, their discovery was poorly timed. By the time they had reached print in 1966, a paper by Lennart Carleson[44] had already been published, establishing the following:

Carleson's Theorem (1966) *The Fourier series of any function in $L^2(\mathbb{T})$ converges almost everywhere.*

This left the headline result of Kahane and Katznelson, as well as that of Katznelson's first paper in the case of $p = 2$, completely redundant before they had even appeared in the public domain. Carleson's result is mentioned in a footnote to the first paper and it is commented that Katznelson and Kahane's work constitutes a "reciprocal" result: that given any set E of measure zero, there exists a continuous function whose Fourier series diverges on E (that is that the hypothesis holds). This establishes that Carleson's result of convergence almost everywhere is the best possible result on $L^2(\mathbb{T})$ and is generally what [76] is cited for today, without any reference to the original statements.

Carleson's paper was certainly a triumph of twentieth century mathematics, finally resolving Luzin's conjecture after over fifty years. In *Mathematical Reviews*[74], Kahane wrote that it was a "découverte spectaculaire" [spectacular discovery] and that "les démonstrations sont très délicates, et forcent l'admiration" [the proofs are very delicate and command admiration]. However, he also concedes that "l'article de l'auteur est très difficile à lire. Il est souhaitable qu'on trouve, ou bien une démonstration plus rapide par une autre méthode, ou bien quelques théorèmes généraux, suggérés par la méthode de l'auteur, d'où les théorèmes sur la convergence des séries de Fourier découleraient sans trop de peine." [The author's article is very difficult to read. It is desirable that either a faster proof by a different method or some general theorems are found, as suggested by the author's method, from where the theorems on the convergence of Fourier series follow without too much difficulty.]

Only a year later, in 1967, Richard Hunt presented a paper at a conference in Illinois, published in 1968[72], extending Carleson's result as follows:

Carleson–Hunt Theorem (1967) *The Fourier series of any function in $L^p(\mathbb{T})$ converges almost everywhere for $p \in (1, \infty)$.*

By the Calderón–Stein result, Carleson's $L^2(\mathbb{T})$ result is completely equivalent to weak $L^2(\mathbb{T})$ boundedness of the maximal operator now generally referred as the Carleson operator:

$$\mathscr{C} f \coloneqq \sup_{N \in \mathbb{N}} \Big| \sum_{|n| \leqslant N} \widehat{f}(n) e^{in\cdot} \Big|.$$

However, it was also well known that by a standard argument, as mentioned in the introduction to this thesis, for any $p \in (1, \infty)$, almost everywhere convergence of the Fourier series of $L^p(\mathbb{T})$ functions follows from weak $L^p(\mathbb{T})$ boundedness of the Carleson operator. As such, Hunt's proof focusses on establishing this boundedness, primarily by following Carleson's method with some adapted definitions. Hunt comments that "the proof of our basic result is essentially the proof of Carleson."[72, p. 236]

Whilst Hunt's paper did not provide it, over the years that followed, Kahane was to get his wish for a more efficient proof of Carleson's Theorem. The first significant development was by Charles Fefferman who published a paper in 1973[57] containing a completely new proof of the $L^2(\mathbb{T})$ result along with an explanation of the modifications to the argument that are necessary to generalise it to establish Hunt's extension. In the introduction to the paper, Fefferman explains that, "unlike Carleson's proof, which makes a careful analysis of the structure of an L^2 function f, our arguments essentially ignore f, and concentrate instead on building up a basic 'partial sum' operator from simpler pieces."

Specifically, Fefferman argues that a bound on the Carleson operator is implied by bounding the operator

$$Tf(x) := \int_{-\pi}^{\pi} \frac{e^{iN(x)y}}{y} f(x-y)\,dy$$

where $N(x)$ is some fixed function of x that the bound is independent of. The crucial concept here is that $N(x)$ replaces a supremum in N and thus linearises the operator under consideration.

Fefferman then fixes an odd smooth function ψ, supported in $[-2\pi, 2\pi]$, such that $\frac{1}{x} = \sum_{j=0}^{\infty} \psi_j(x)$ for $x \in [-\pi, \pi]$, where $\psi_j = 2^j \psi(2^j \cdot)$. The fundamental idea behind the "simpler pieces" is that they are parameterised by pairs of dyadic intervals, $p = [\omega, I]$, where $|\omega||I| = 1$, $\omega \subseteq \mathbb{R}$, $I \subseteq [0, 2\pi]$ and, loosely speaking, the pieces are localised to I whilst their Fourier transforms are localised to ω. This concept of simultaneous localisation is the basic idea behind so-called time-frequency (or wave packet) analysis.

More precisely stated, the "piece" of T associated to a pair $p = [\omega, I]$ where $|I| = 2^{-k}$ is given by

$$T_{[\omega,I]} f(x) = ((e^{iN(x)\cdot}\psi_k) * f)(x)\chi_{E(\omega,I)}(x)$$

where $E(\omega, I) = \{x \in I : N(x) \in \omega\}$. The proof then proceeds by grouping the pairs p into collections with suitably "nice" properties, allowing the desired bound to be established.

In 1980, a paper by Carlos Kenig and Peter Tomas[80] on transference* of maximal operators was published. Circumstances under which functions that were Fourier multipliers on $L^p(\mathbb{T})$ were also Fourier multipliers on $L^p(\mathbb{R})$ (and vice versa) were already known from a 1965 paper of Karel de Leeuw[96]. Kenig and Tomas extended the scope of de Leeuw's result, proving the following:

*They use the term "transplantation".

Theorem (Kenig and Tomas, 1980) *Let $m \in L^\infty(\mathbb{R}^n)$ be such that every $x \in \mathbb{R}^n$ is a Lebesgue point of m and for each $R > 0$, define $m_R := m\left(\frac{\cdot}{R}\right)$. Fix any $p \in (1, \infty)$ and let $T_{\mathbb{R}^n}^{m_R}$ and $T_{\mathbb{T}^n}^{m_R}$ be the operators with Fourier multiplier m_R acting on functions on $\mathbb{R}^n$ and $\mathbb{T}^n$ respectively. Then*

- $\sup_{R>0} |T_{\mathbb{R}^n}^{m_R} \cdot|$ *is bounded on $L^p(\mathbb{R}^n)$ if and only if* $\sup_{R>0} |T_{\mathbb{T}^n}^{m_R} \cdot|$ *is bounded on $L^p(\mathbb{T}^n)$;*
- $\sup_{R>0} |T_{\mathbb{R}^n}^{m_R} \cdot|$ *is weakly bounded on $L^p(\mathbb{R}^n)$ if and only if* $\sup_{R>0} |T_{\mathbb{T}^n}^{m_R} \cdot|$ *is weakly bounded on $L^p(\mathbb{T}^n)$.*

This result has the consequence that weak boundedness of the Carleson operator on $L^p(\mathbb{T})$ is equivalent to the corresponding result for the Carleson operator for Fourier integrals on $L^p(\mathbb{R})$, namely that the operator

$$\mathscr{C}_{\mathbb{R}} f := \sup_{N \in \mathbb{R}^+} \left| \int_{|\xi| \leqslant N} \widehat{f}(\xi) e^{2\pi i \xi x} \, d\xi \right|$$

is weakly bounded on $L^p(\mathbb{R})$. The validity of this boundedness also establishes convergence of Fourier integrals on $L^p(\mathbb{R})$, that is to say that for any $f \in L^p(\mathbb{R})$,

$$\lim_{N \to \infty} \int_{|\xi| \leqslant N} \widehat{f}(\xi) e^{2\pi i \xi x} \, d\xi = f(x)$$

for almost every $x \in \mathbb{R}$.

By the end of the twentieth century, being able to study boundedness of the Carleson operator on $\mathbb{R}$ rather than $\mathbb{T}$ became an advantage as research activity in harmonic analysis on $\mathbb{R}^n$ was becoming vastly more commonplace than that on $\mathbb{T}^n$. By the time of the publication of the paper of Kenig and Tomas, the techniques of analysis on $\mathbb{R}^n$ had already become much better developed than those on $\mathbb{T}^n$.

It was by considering the Carleson operator on $\mathbb{R}$ that Michael Lacey and Christoph Thiele provided the simplest known proof of Carleson's Theorem to date in a paper published in

2000[92]. The proof has its origins in their earlier work on the boundedness of the bilinear Hilbert transform, where the time-frequency analytic ideas of Fefferman and Carleson served as inspiration. Their development of these techniques won them the Salem prize in 1996[26] and their first paper of several on the topic of the bilinear Hilbert transform, establishing boundedness on $L^p(\mathbb{R})$ for $p \in (2,\infty)$, was published in 1997[91].

In [92], Lacey and Thiele show weak boundedness of the Carleson operator on $L^2(\mathbb{R})$ using an approach strongly influenced by Fefferman's paper. Their methods provide the underpinning for the first part of this thesis and thus the proof will be discussed in some detail in Section 1.2.

In 2004, a paper by Loukas Grafakos, Terence Tao and Erin Terwilleger[68] was published, providing a new proof of Hunt's extension of Carleson's Theorem, establishing boundedness of the Carleson operator on $L^p(\mathbb{R})$ for $p \in (1,\infty)$. Their method is based on a variation of Lacey and Thiele's techniques for $L^2(\mathbb{R})$ combined with an extension of a result of Terwilleger together with Malabika Pramanik[119], also based on Lacey and Thiele's methods. According to Grafakos, Tao and Terwilleger, whilst they never published it, Lacey and Thiele also managed to provide a generalised version of their original proof to establish the same result, although it was "rather complicated compared with the relatively short and elegant proof they gave for $p = 2$."[68, p. 322]

Aside from $L^p(\mathbb{R})$ and $L^p(\mathbb{T})$ boundedness of the Carleson operator, there have been a number of other extensions of Carleson's original result. As stated earlier, due to Kolmogorov's example, it is known that there exist functions with everywhere divergent Fourier series in $L^1(\mathbb{T})$, so in light of Hunt's result, asking whether almost everywhere convergence holds in Orlicz spaces "near" to L^1 is a pertinent question. In 1971, Per Sjölin established that this is the case for the space $L(\log L)(\log\log L)$[128] and in 1995, Nikolai Antonov managed to improve this to the space $L(\log L)(\log\log\log L)$[4]. Whilst the question of convergence for $L(\log L)$ remains open, in 1981, Thomas Körner established failure of Carleson's result for

$L(\log\log L)$[88] and in 2000, Sergei Konyagin improved this[86], managing to establish that if $\phi(x) = o\left(\dfrac{x\sqrt{\log x}}{\sqrt{\log\log x}}\right)$ as $x \to \infty$, then the result fails on the space $\phi(L)$.

Recently, Richard Oberlin, Andreas Seeger, Terence Tao, Christoph Thiele and James Wright have considered an alternative generalisation of Carleson's Theorem[116], replacing the Carleson operator with a strictly larger operator and proving its L^p boundedness for certain values of p. Their method is based on that of Lacey and Thiele[92] and its refinement by Grafakos, Tao and Terwilleger[68]. Specifically, they consider the q-variational norm, defined for a family of complex numbers (a_ξ) and $q \in [1,\infty)$ as

$$\|(a_\xi)\|_{V_\xi^q} := \sup_{K\in\mathbb{N}} \sup_{\substack{(\xi_k)_{k=0}^K \subseteq \mathbb{C} \\ \xi_0 < \cdots < \xi_K}} \left(\sum_{k=1}^{K} |a_{\xi_k} - a_{\xi_{k-1}}|^q\right)^{\frac{1}{q}}$$

and show L^p boundedness of the q-variational Carleson operator,

$$\mathscr{C}_{V^q} f := \left\| \int_{-\infty}^{\xi} \widehat{f}(\lambda) e^{2\pi i \lambda \cdot}\, d\lambda \right\|_{V_\xi^q},$$

for $q > 2$, $p \in (q', \infty)$, where q' is the conjugate exponent of q, satisfying $\frac{1}{q} + \frac{1}{q'} = 1$. This result, which includes the Carleson–Hunt theorem, has the advantage of providing quantitative information on the rate of convergence of Fourier series and integrals.

1.2 Time-Frequency Analysis and Boundedness of the Carleson Operator on $L^2(\mathbb{R})$

As previously mentioned, the time-frequency methods in this thesis are based on those used by Lacey and Thiele in [92] in their proof of Carleson's Theorem. This section will provide an overview of these methods. The exposition is based on the author's unpublished lecture notes[12] as well as [92], [90] and [67, Ch. 11]; see also [142].

1.2.1 Localisation to Tiles and the Model Operator

As discussed in the previous section, in their proof of Carleson's Theorem, Lacey and Thiele consider the analogous and equivalent problem of convergence of Fourier integrals. They define the one-sided maximal operator,

$$\mathscr{C} f := \sup_{N \in \mathbb{R}} \left| \int_{-\infty}^{N} \widehat{f}(\xi) e^{2\pi i \xi \cdot} \, d\xi \right|,$$

acting on functions $f \in L^2(\mathbb{R})$ and set out to establish its boundedness as an operator from $L^2(\mathbb{R})$ to $L^{2,\infty}(\mathbb{R})$. In a similar spirit to Fefferman's proof of Carleson's Theorem, their approach is based on the idea of splitting the Fourier partial summation operator into pieces that, in some sense, localise both the operator (applied to a given function) and its Fourier transform. Once this has been achieved, the pieces are separated into collections according to the values of certain quantities expressing properties of each piece. These collections are handled individually and the results are then combined to provide a global estimate.

The so-called "pieces" are parameterised by tiles in $\mathbb{R}^2$, which is thought of as being the "time-frequency plane".* The projection of a tile s onto the first coordinate axis is denoted by I_s and represents the "time localisation" that will be applied to the operator (that is where the operator is concentrated), whilst the projection onto the second coordinate axis is denoted by ω_s and represents the "frequency localisation" that will be applied (that is where the Fourier transform of the operator is concentrated). All tiles are assumed to have area 1 and to be dyadic in the sense that $I_s = [l2^k, (l+1)2^k)$ and $\omega_s = [m2^{-k}, (m+1)2^{-k})$ for some $k, l, m \in \mathbb{Z}$. The collection of all dyadic tiles is denoted by $\mathbb{D}$ and the collection of all dyadic tiles at a single scale, $k \in \mathbb{Z}$, is denoted by $\mathbb{D}_k := \{s \in \mathbb{D} : |I_s| = 2^k\}$. Further, the frequency projection of any given tile will be split into its upper and lower halves, denoted by ω_s^+ and ω_s^- respectively.

*These tiles are the same objects as Fefferman's "pairs".

Throughout this thesis, for an interval J, the notation $c(J)$ will be used to denote its centre. For a positive constant α, αJ will be used to denote the set $\{\alpha x : x \in J\}$ whilst $\alpha \star J$ will be used to denote the interval with the same centre, but with $|\alpha \star J| = \alpha |J|$.

The concept of "localisation" of the Fourier partial summation operator to a tile s is slightly more technical than might be desired. The ideal localisation of an operator would be to compactly support it and its Fourier transform in (subsets of) the appropriate intervals associated to s. However, it is a consequence of the uncertainty principle that this is not possible. Instead, the localisation will be such as to compactly support the Fourier transform of the operator in some subset of ω_s and ensure rapid decay of the operator itself away from I_s.

More specifically, fix $\phi \in \mathscr{S}(\mathbb{R})$ such that $\widehat{\phi}(\xi) \in \mathbb{R}_0^+$ for any $\xi \in \mathbb{R}$ and so that $\operatorname{supp}(\widehat{\phi}) \subseteq [-\frac{1}{20}, \frac{1}{20}]$. For each $s \in \mathbb{D}$, define

$$\phi_s(x) = |I_s|^{-\frac{1}{2}} \phi\left(\frac{x - c(I_s)}{|I_s|}\right) e^{2\pi i c(\omega_s^-) x}.$$

It can be seen by straightforward calculation (using that $|I_s||\omega_s| = 1$) that

$$\widehat{\phi_s}(\xi) = |\omega_s|^{-\frac{1}{2}} \widehat{\phi}\left(\frac{\xi - c(\omega_s^-)}{|\omega_s|}\right) e^{2\pi i (c(\omega_s^-) - \xi) c(I_s)}.$$

Each $\widehat{\phi_s}$ is supported in $\frac{1}{5} \star \omega_s^-$ whilst ϕ_s is "roughly supported" in I_s in the sense that it decays rapidly away from I_s, as ϕ is a Schwartz function. The restriction of the support of $\widehat{\phi_s}$ to only part of ω_s^- is done for technical reasons. However, to aid intuition at this stage, $\widehat{\phi_s}$ will be thought of as simply "localised" to the whole of ω_s^-.

A model operator for Fourier summation can be defined in terms of this localising function as follows:

$$A_\xi f = \sum_{s \in \mathbb{D}} \chi_{\omega_s^+}(\xi) \langle f, \phi_s \rangle \phi_s.$$

The inner product here is taken in the sense of $L^2(\mathbb{R})$. This operator can also be considered at individual scales by defining

$$A_\xi^k f = \sum_{s \in \mathbb{D}_k} \chi_{\omega_s^+}(\xi) \langle f, \phi_s \rangle \phi_s$$

for some $k \in \mathbb{Z}$.

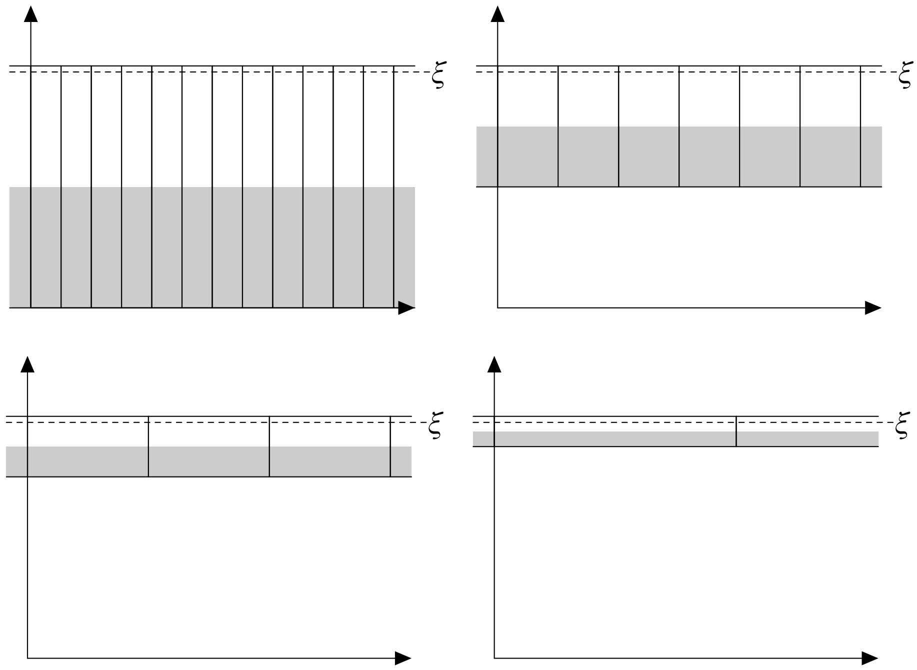

Figure 1.1 – The model operator A_ξ^k in the time-frequency plane at different scales, k.

Roughly speaking, A_ξ takes all tiles with ξ in their upper half and sums the parts of f localised to the lower half of these tiles. Figure 1.1 illustrates this procedure for tiles at individual fixed scales (that is to say that it illustrates A_ξ^k for various choices of k), with the parts of f that are summed being localised to the shaded regions. It is clear from examining the diagrams that wider scales sum parts of f with $\widehat{f}$ close to ξ, whilst taller scales sum parts with $\widehat{f}$ far away from ξ. Thus, in a sense, $A_\xi f$ acts as a model for $\int_{-\infty}^{\xi} \widehat{f}(\lambda) e^{2\pi i \lambda \cdot} \, d\lambda$, the Fourier partial sum operator and hence $\sup_{\xi \in \mathbb{R}} |A_\xi f|$ (the "maximal model operator") acts as a model for $\mathscr{C} f$.

The analogies between $\sup_{\xi \in \mathbb{R}} |A_\xi f|$ and $\mathscr{C} f$ can be further examined by considering the various symmetry properties that each operator has. Define the following three operations of modulation, translation and dilation on functions $f \in L^p(\mathbb{R})$, $p \in [1,\infty]$ for $a \in \mathbb{R}$:

$$M_a f(x) := f(x)e^{2\pi i a x}, \quad \tau_a f(x) := f(x-a), \quad D_a f(x) := f(a^{-1}x).$$

The Carleson operator is modulation invariant in the sense that $\mathscr{C} M_\eta f = \mathscr{C} f$ for any $\eta \in \mathbb{R}$. Indeed, it is precisely this modulation invariance that makes studying the Carleson operator "difficult" as many of the standard techniques in harmonic analysis do not possess a similar invariance and thus are largely inapplicable.* At individual scales, the maximal model operator also has a certain modulation invariance in the sense that $\sup_{\xi\in\mathbb{R}} |A_\xi^k M_{m2^{-k}} f| = \sup_{\xi\in\mathbb{R}} |A_\xi^k f|$ for any $m \in \mathbb{Z}$. Modulation of a function corresponds to translation of its Fourier transform and so is equivalent to vertical movement in the time-frequency plane. This equation hence expresses the fact that, at a single scale, "moving a function vertically" can be compensated for by "moving" the ξ-line vertically by the same amount, so long as the movement corresponds to a multiple of the height of the tiles.

The Carleson operator also commutes with translations in the sense that $\tau_{-y} \mathscr{C} \tau_y f = \mathscr{C} f$ for any $y \in \mathbb{R}$. Whilst this is a property that is better understood in classical harmonic analysis than modulation invariance, a suitable model should express a similar symmetry. Considering individual scales of the maximal model operator again, it can be said that $\sup_{\xi\in\mathbb{R}} |\tau_{-m2^k} A_\xi^k \tau_{m2^k} f| = \sup_{\xi\in\mathbb{R}} |A_\xi^k f|$ for any $m \in \mathbb{Z}$. Translation equates to horizontal movement in the time-frequency plane and so this equation expresses the fact that, at a single scale, "moving a function horizontally" can be compensated for by "moving" the tiles horizontally by the same amount, so long as the movement corresponds to a multiple of the width of the tiles.

Finally, the Carleson operator also commutes with dilations: $D_{a^{-1}} \mathscr{C} D_a f = \mathscr{C} f$ for any $a \in \mathbb{R}$. Dilation of a function effects a reciprocal dilation on its Fourier transform, so in the case of the maximal model operator, dilation conjugation causes a change of scale in the sense

*Modulation invariance is a property that the Carleson operator and the bilinear Hilbert transform share and it is because of this that Lacey and Thiele's works on the two topics have a common heritage.

that $\sup_{\xi\in\mathbb{R}} |D_{2^{-l}} A_\xi^k D_{2^l} f| = \sup_{\xi\in\mathbb{R}} |A_\xi^{k+l} f|$ for any $l \in \mathbb{Z}$. Thinking in terms of the time-frequency picture again, dilating a function can be compensated for by applying the inverse dilation to the ξ-line and shifting the scale of the tiles appropriately, resulting in what are effectively dilated versions of the original tiles. Since the full model operator sums all scales, it can be said that $\sup_{\xi\in\mathbb{R}} |D_{2^{-l}} A_\xi D_{2^l} f| = \sup_{\xi\in\mathbb{R}} |A_\xi f|$ for any $l \in \mathbb{Z}$.

Whilst these quasi-symmetries demonstrate some analogies between the Carleson operator and the maximal model operator, they are not sufficient on their own to establish an equivalence of the desired form. The solution to the problem of improving these properties is to consider a new operator that averages the model operator conjugated with modulations, translations and dilations. Specifically, define the operator

$$\Pi_\xi f \coloneqq \lim_{K,L\to\infty} \frac{1}{4KL} \int_{-L}^{L} \int_{-K}^{K} \int_0^1 M_{-\eta} \tau_{-y} D_{2^{-\kappa}} A_{2^{-\kappa}(\xi+\eta)} D_{2^\kappa} \tau_y M_\eta f \, d\kappa \, dy \, d\eta.$$

Note that the parameter $2^{-\kappa}(\xi+\eta)$ on the model operator in the integrand is due to the movements of the "ξ-line" described in the above discussion of the modulation and dilation quasi-symmetries.

This new averaged operator, Π_ξ, has some genuine symmetries that can be shown to imply that $\mathscr{C} f \approx \sup_{\xi\in\mathbb{R}} |\Pi_\xi f|$. It follows that to show weak L^2 boundedness of the Carleson operator, it suffices to show weak L^2 boundedness of this maximal averaged operator. Further, it can be established that weak boundedness of the maximal averaged operator follows from that of the maximal model operator, so in fact, it does suffice to show weak boundedness of $\sup_{\xi\in\mathbb{R}} |A_\xi f|$ over $f \in L^2(\mathbb{R})$.

This maximal operator can be linearised. Indeed, for any fixed f, a measurable function $N_f : \mathbb{R} \to \mathbb{R}$ can be selected that chooses values of ξ where the supremum is essentially attained, in the sense that $\sup_{\xi\in\mathbb{R}} |A_\xi f(x)| \leqslant 2|A_{N_f(x)} f(x)|$. With this in mind, it suffices to choose an arbitrary measurable function $N : \mathbb{R} \to \mathbb{R}$ and bound $A_N f$ with constant independent of the choice of N. Further, to simplify calculations, the sum over $\mathbb{D}$ in the definition of A_N can be replaced with a sum over some arbitrary finite subcollection $\mathbb{P} \subseteq \mathbb{D}$.

These reductions given, it is seen to suffice to bound the operator

$$A_{N,\mathbb{P}} f(x) := \sum_{s \in \mathbb{P}} \chi_{\omega_s^+}(N(x)) \langle f, \phi_s \rangle \phi_s(x)$$

from $L^2(\mathbb{R})$ to $L^{2,\infty}(\mathbb{R})$ with constant independent of N and $\mathbb{P}$, that is to say show that

$$|\{x \in \mathbb{R} : |A_{N,\mathbb{P}} f(x)| > t\}| \lesssim \left(\frac{\|f\|_{L^2(\mathbb{R})}}{t} \right)^2$$

for all $t \in \mathbb{R}^+$. This is implied by the estimate

$$\int_E |A_{N,\mathbb{P}} f(x)|\, dx \lesssim |E|^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})}$$

for any measurable set $E \subseteq \mathbb{R}$ of finite measure by setting $E = \{x \in \mathbb{R} : |A_{N,\mathbb{P}} f(x)| > t\}$.

Now, $\int_E |A_{N,\mathbb{P}} f(x)|\, dx$ can be replaced with $\left| \int_E A_{N,\mathbb{P}} f(x)\, dx \right|$ by making a trivial decomposition of E. Rearranging the resultant inequality, this reduces proving Carleson's Theorem to establishing the following main estimate:

$$\boxed{\left| \sum_{s \in \mathbb{P}} \langle (\chi_{\omega_{s+}} \circ N) \phi_s, \chi_E \rangle \langle f, \phi_s \rangle \right| \lesssim |E|^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})}}$$

for any measurable set $E \subseteq \mathbb{R}$, measurable function $N : \mathbb{R} \to \mathbb{R}$ and subcollection $\mathbb{P} \subseteq \mathbb{D}$, with constant independent of E, N and $\mathbb{P}$.

1.2.2 The Selection Process for Grouping Tiles

To allow for tiles to be grouped, a partial ordering on tiles, $<$, is defined: for tiles s, $s' \in \mathbb{D}$, it is said that $s < s'$ if $I_s \subseteq I_{s'}$ and $\omega_{s'} \subseteq \omega_s$. A collection of tiles, $\mathbb{T} \subseteq \mathbb{D}$, is said to be a tree if it has a maximal element under $<$, referred to as the "top" of $\mathbb{T}$ and written as $\mathrm{top}(\mathbb{T})$.

Note that owing to the dyadic sizing and positioning of tiles, if two tiles, s and s', intersect, necessarily either $s < s'$ or $s' < s$. It follows that in any given collection, distinct tiles that are maximal under $<$ are disjoint.*

Since the method of localisation of functions to tiles in the model operator gives particular roles to the lower and upper halves of tiles in the time-frequency plane, in addition to the concept of a tree, it is useful to have notions of tile grouping that reflect this division. A tree $\mathbb{T}$ is said to be a +tree if $\omega^+_{\mathrm{top}(\mathbb{T})} \subseteq \omega^+_s$ for any $s \in \mathbb{T}$. Analogously, it is said to be a –tree if $\omega^-_{\mathrm{top}(\mathbb{T})} \subseteq \omega^-_s$ for any $s \in \mathbb{T}$. Any tree can be written uniquely as the union of a +tree and a –tree which share the same top but are otherwise disjoint.

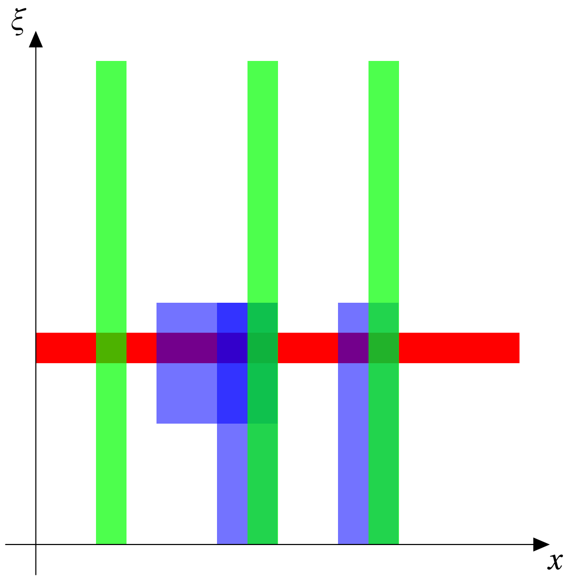

Figure 1.2 – A tree in the time-frequency plane. Its top is the red tile which together with the blue tiles forms a +tree and together with the green tiles forms a –tree.

To establish the main estimate, two quantities are defined to measure properties of the collections of tiles involved. The first of these is the mass of a set E with respect to a tile s, defined as follows:

$$\mathcal{M}(E;\{s\}) := |E|^{-1} \sup_{\substack{u\in\mathbb{D}\\ s<u}} \int_{E\cap N^{-1}(\omega_u)} \frac{|I_u|^{-1}}{(1+\frac{|x-c(I_u)|}{|I_u|})^{10}}\, dx.$$

*It is noted that it is possible for $s < s'$ and $s' < s$ to hold simultaneously (when $s = s'$). Nonetheless, the symbol $<$ is used rather than $\leqslant$ as this is standard in the literature.

Considering the left-hand side of the main estimate, $\left|\sum_{s\in\mathbb{P}}\langle(\chi_{\omega_s^+}\circ N)\phi_s,\chi_E\rangle\langle f,\phi_s\rangle\right|$, it can be seen that $\mathscr{M}(E;\{s\})$ is a generalisation of a natural expression of density of E with respect to a tile s, namely $\frac{|I_s\cap E\cap N^{-1}(\omega_s^+)|}{|E||I_s|}$. Replacing χ_{I_s} with a decay function is a reflection on the fact that ϕ_s is only "well-localised" to I_s, not supported there. The reasons for the introduction of the supremum and the replacement of $N^{-1}(\omega_u^+)$ with $N^{-1}(\omega_u)$ are less evident, but these changes are requisite to the proof of one of the lemmata involved in proving the main estimate.

Generalising this definition of the mass of E with respect to an individual tile, the mass of E with respect to an arbitrary collection of tiles $\mathbb{S}$ is given as follows:

$$\mathscr{M}(E;\mathbb{S}) \coloneqq \sup_{s\in\mathbb{S}}\mathscr{M}(E;\{s\}).$$

Lacey and Thiele use the term "mass" rather than "density" as they assume that $|E|=1$ in their presentation of the proof. This will not be done here in order to make the true role of this quantity more apparent. However, the name "mass" is retained to avoid confusion.

The other "measuring quantity" involved in Lacey and Thiele's proof of Carleson's Theorem is the energy of a function f with respect to a collection of tiles $\mathbb{S}$, defined as follows:

$$\mathscr{E}(f;\mathbb{S}) = \frac{1}{\|f\|_{L^2(\mathbb{R})}}\sup_{\substack{\mathbb{T}\text{ a +tree}\\ \mathbb{T}\subseteq\mathbb{S}}}\left(\frac{1}{|I_{\mathrm{top}(\mathbb{T})}|}\sum_{s\in\mathbb{T}}|\langle f,\phi_s\rangle|^2\right)^{\frac{1}{2}}.$$

If $\mathbb{S}$ consists of a single tile only, this expression looks like a very natural candidate for the density of the "amount" of the function f that is concentrated to the lower half of s. For more general $\mathbb{S}$, the supremum over +trees is sensible as the lower halves of tiles (which are where the Fourier transforms of the ϕ_s are localised) in +trees are necessarily disjoint.

Analogously to "mass", Lacey and Thiele's term, "energy", will be used here rather than "energy density", despite the fact that this presentation will not assume that $\|f\|_{L^2(\mathbb{R})}=1$.

The bulk of Lacey and Thiele's paper[92] consists of proving the following three lemmata:

The Mass Lemma* *Let $E \subseteq \mathbb{R}$ be a measurable set of finite measure, let $N : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function and let $\mathbb{P}$ be an arbitrary finite collection of dyadic tiles. Then $\mathbb{P}$ can be written as $\mathbb{P} = \mathbb{P}^{\text{light}} \sqcup \mathbb{P}^{\text{heavy}}$, where*

$$\mathscr{M}(E; \mathbb{P}^{\text{light}}) \leqslant \frac{1}{4}\mathscr{M}(E; \mathbb{P})$$

and $\mathbb{P}^{\text{heavy}}$ is a union of trees $\mathbb{T}_j$ with

$$\sum_j |I_{\text{top}(\mathbb{T}_j)}| \lesssim \mathscr{M}(E; \mathbb{P})^{-1}.$$

The Energy Lemma *Take $f \in L^2(\mathbb{R})$ and let $\mathbb{P}$ be an arbitrary finite collection of dyadic tiles. Then $\mathbb{P}$ can be written as $\mathbb{P} = \mathbb{P}^{\text{low}} \sqcup \mathbb{P}^{\text{high}}$, where*

$$\mathscr{E}(f; \mathbb{P}^{\text{low}}) \leqslant \frac{1}{2}\mathscr{E}(f; \mathbb{P})$$

and $\mathbb{P}^{\text{high}}$ is a union of trees $\mathbb{T}_j$ with

$$\sum_j |I_{\text{top}(\mathbb{T}_j)}| \lesssim \mathscr{E}(f; \mathbb{P})^{-2}.$$

The Tree Lemma *Take $f \in L^2(\mathbb{R})$, let $E \subseteq \mathbb{R}$ be a measurable set of finite measure, let $N : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function and let $\mathbb{T} \subseteq \mathbb{D}$ be a tree. Then*

$$\sum_{s \in \mathbb{T}} |\langle (\chi_{\omega_{s+}} \circ N)\phi_s, \chi_E \rangle \langle f, \phi_s \rangle| \lesssim |I_{\text{top}(\mathbb{T})}| \mathscr{E}(f; \mathbb{T}) \mathscr{M}(E; \mathbb{T}) \|f\|_{L^2(\mathbb{R})} |E|.$$

*In [92], the first inequality here is stated with the constant $\frac{1}{2}$ instead of $\frac{1}{4}$. The choice of this constant is unimportant, but $\frac{1}{4}$ seems slightly more natural to the present author in the context of the application of this lemma.

Using an inductive argument, the mass and energy lemmata allow $\mathbb{P}$ to be written as the disjoint union $\bigsqcup_{n=-\infty}^{n_0} \mathbb{P}_n$ for some $n_0 \in \mathbb{Z}$ where for each n,

$$\mathcal{M}(E;\mathbb{P}_n) \leqslant 2^{2n}, \qquad \mathcal{E}(f;\mathbb{P}_n) \leqslant 2^n,$$

and each $\mathbb{P}_n$ is a union of trees $\mathbb{T}_{n_j}$ such that

$$\sum_j |I_{\mathrm{top}(\mathbb{T}_{n_j})}| \lesssim 2^{-2n}.$$

The sum over $\mathbb{P}$ from the main estimate can then be broken up into the constituent trees. The tree lemma allows each tree to be handled individually, whilst the bounds on the mass, energy and sizes of the tops of the trees ensure that the resulting quantity is summable, allowing the desired bound to be established, completing the proof of Carleson's Theorem.

Lacey and Thiele's proof of the mass lemma begins by forming the collection $\mathbb{P}^{\text{heavy}}$ by selecting tiles that individually have large mass with respect to the set E. This allows $\mathbb{P}^{\text{light}}$, the collection of all remaining tiles, to satisfy the desired bound on its mass automatically. Since $\mathbb{P}^{\text{heavy}}$ can be grouped into trees, $\mathbb{T}_j$, by choosing its maximal elements (with respect to the partial order $<$) as the tops, the main task in the proof is showing that $\sum_j |I_{\mathrm{top}(\mathbb{T}_j)}| \lesssim \mathcal{M}(E;\mathbb{P})^{-1}$ holds.

The proof of the energy lemma in [92] follows a similar, albeit slightly more sophisticated, scheme to that of the mass lemma. The collection $\mathbb{P}^{\text{high}}$ is formed by employing a specific algorithm to select +trees, $\mathbb{T}_j+$, from $\mathbb{P}$ that are of high energy, in each case extending these +trees to the regular trees, $\mathbb{T}_j$, by selecting as many tiles as possible, $s \in \mathbb{P}$, that satisfy $s < \mathrm{top}(\mathbb{T}_j+)$. Since this ensures that the collection of remaining tiles in $\mathbb{P}$, designated $\mathbb{P}^{\text{low}}$, automatically satisfies the desired bound on its energy, the main task of the proof is showing that $\sum_j |I_{\mathrm{top}(\mathbb{T}_j)}| \lesssim \mathcal{E}(f;\mathbb{P})^{-2}$ holds.

The proofs of the remaining estimates on the sizes of the tops of the trees in the mass and energy lemmata proceed by using geometric and combinatorial arguments in the time-frequency plane, also employing estimates for the localising function, ϕ_s, in the case of the energy lemma.

In their proof of the tree lemma, Lacey and Thiele rewrite the desired inequality in terms of a sum of localised L^1 norms, allowing them to separately estimate terms where the domain of integration is "away" from and "near" to the time projections of the tiles of the tree. Since the bulk of the information carried by the left hand side of the inequality in the tree lemma is localised to the time projections of the tiles that it is made up of, the "away" term is the easier of the two terms to bound. Establishing the appropriate estimates on both terms requires geometric and combinatorial arguments in the time-frequency plane, as in the proofs of the mass and energy lemmata, but the arguments employed to bound the "near" term are significantly more intricate.

1.3 Almost Periodic Functions

1.3.1 Standard Theory

The aim of this section is to provide a concise introduction to the standard theory of almost periodic functions. It is based upon the first chapter of the author's previous thesis[11], although, in the interests of avoiding too much repetition, it will be somewhat briefer. Proofs will only be provided for results that were not stated previously. For further details, the reader is referred to the earlier thesis and the references which it draws on (principally [97], [17], [21] and [3], but see also [49] and [98]).

The original definition of almost periodicity was given by Harald Bohr in the 1920s and is as follows:

Definition 1.3.1 (Bohr Almost Periodicity) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then f is said to be almost periodic if for all $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that for any $x_0 \in \mathbb{R}$, there exists $\tau \in [x_0, x_0 + K_\varepsilon]$ satisfying $\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| < \varepsilon$.*

Loosely speaking, this says that for any prescribed degree of accuracy ε, there are translation numbers τ, well-distributed across $\mathbb{R}$, such that the function f repeats to within ε when translated by τ. Grouping these functions together and equipping them with the supremum norm results in a Banach space, referred to as the Bohr class, B.

Almost periodic trigonometric polynomials are a natural generalisation of periodic trigonometric polynomials:

Definition 1.3.2 (Trigonometric Polynomials) *The set $\mathscr{P}$ of almost periodic trigonometric polynomials is defined to be the collection of all functions f of the form*

$$f(x) = \sum_{n=-N}^{N} a_n e^{i\lambda_n x}$$

where $(a_n)_{n=-N}^{N} \subset \mathbb{C}$, $(\lambda_n)_{n=-N}^{N} \subset \mathbb{R}$ and $N \in \mathbb{N}$.

Significantly, the Bohr class satisfies the following "Fundamental Theorem":

Theorem 1.3.3 (The Fundamental Theorem) *The Bohr class, B, is identically equal to the closure of the set of trigonometric polynomials, $\mathscr{P}$, in the space $C(\mathbb{R})$ equipped with the supremum norm.*

There are a number of different generalisations of Bohr almost periodicity. The spaces of relevance in this thesis are the Besicovitch spaces, B^p, the most general of the classical almost periodic functions spaces (Bohr, Stepanov, Weyl and Besicovitch). A definition of these spaces of a similar form to Definition 1.3.1 can be formulated, but the condition for τ being well-distributed is more complicated.* An equivalent, simpler definition, and one that will be entirely adequate here, is the following:

*Note that in the author's previous thesis[11], the Besicovitch spaces were defined with the same well-distributed condition as in Definition 1.3.1. This is a genuine almost periodic function space, strictly larger than the one that will be considered here, but does not satisfy the Fundamental Theorem as is stated there. For further details see the erratum notice.

Definition 1.3.4 (Besicovitch Almost Periodicity) *For any fixed $p \in [1,\infty)$, the Besicovitch (semi-)norm, acting on functions $f \in L^p_{\text{loc}}(\mathbb{R})$ is defined as*

$$\|f\|_{B^p} := \lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}|f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

The Besicovitch space of almost periodic functions, B^p, is defined to be the closure of the set of trigonometric polynomials, $\mathscr{P}$, with respect to this norm.

The limit in the definition of the Besicovitch norm can be shown to exist for any Besicovitch almost periodic function.

If functions that are equivalent under the Besicovitch norm are considered to be equal then these spaces are Banach spaces and for $p_1, p_2 \in [1,\infty)$ with $p_1 < p_2$, it can be shown that $B \subset B^{p_2} \subset B^{p_1}$. It follows that B^1 will be the largest class of functions under consideration. It should be noted that functions that differ on sets of positive or even infinite measure can still be equivalent.

In addition to the Besicovitch norm, there is also a natural (*bona fide*) inner product on B^2 given by the following expression:

$$\langle\!\langle f, g\rangle\!\rangle := \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)\overline{g(x)}\,dx.$$

The unusual notation here is introduced to allow easy distinction from other inner products later.

It is very important to clarify that $B^p \neq \{f \in L^p_{\text{loc}}(\mathbb{R}) : \|f\|_{B^p} < \infty\}$. The latter space is a strict superspace of the former.

The Fourier series of an almost periodic function f is given by*

$$\sum_{n\in\mathbb{Z}} \widehat{f}(\lambda_n)e^{2\pi i\lambda_n\cdot}$$

*The present definition represents a renormalisation of the definition given in [11]. This has been done to streamline certain interactions between Fourier transforms and almost periodic Fourier series that will occur later. It is easy to see that the two definitions are entirely equivalent.

where

$$\widehat{f}(\lambda_n) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-2\pi i \lambda_n x}\, dx$$

and λ_n ranges over the necessarily countable set

$$\sigma(f) := \{\lambda \in \mathbb{R} : \widehat{f}(\lambda) \neq 0\}.$$

This set will be referred to as the spectrum of f.

As a matter of convention, the λ_n will be ordered to be strictly increasing and so that $\lambda_{-n} = -\lambda_n$ (permitting $\widehat{f}(\lambda_{-n})$ to be zero in this case where necessary). The Fourier series of B^1 functions are unique in the sense that two inequivalent functions cannot have the same expansions.

The following is a natural condition to impose on almost periodic functions:

Definition 1.3.5 (Separation Condition) *A function $f \in B^1$ will be said to satisfy the separation condition if there exists a positive separation constant α_f such that for any distinct λ, $\gamma \in \sigma(f)$, it is the case that $|\lambda - \gamma| \geqslant \alpha_f$.*

Throughout this thesis, if an almost periodic function f satisfies the separation condition, α_f will be used to denote the supremum of all possible separation constants.

The Besicovitch averaging operation used to calculate the Fourier coefficients of an almost periodic function has a certain translation invariance to it, as expressed by the following result:

Theorem 1.3.6 *Let $f \in B^1$. Then for any $a \in \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T+a}^{T+a} f(x)\, dx = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx.$$

Proof For a real-valued function h, define $h^{\pm} \coloneqq \pm\chi_{\{\pm h>0\}}h$. It can be seen that $\mathrm{Re}(f)^{\pm}$, $\mathrm{Im}(f)^{\pm} \in B^1$, so

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)\,dx &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{Re}(f)^{+}(x)\,dx - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{Re}(f)^{-}(x)\,dx \\ &\quad + i\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{Im}(f)^{+}(x)\,dx - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{Im}(f)^{-}(x)\,dx\right).\end{aligned}$$

As such, it may be assumed without loss of generality that f is real-valued and non-negative. This given, consider that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a}f(x)\,dx &\leqslant \lim_{T\to\infty}\frac{1}{2T}\int_{-(T+|a|)}^{T+|a|}f(x)\,dx \\ &= \lim_{T\to\infty}\frac{T+|a|}{T}\frac{1}{2(T+|a|)}\int_{-(T+|a|)}^{T+|a|}f(x)\,dx \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)\,dx.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)\,dx &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T-a+a}^{T-a+a}f(x)\,dx \\ &\leqslant \lim_{T\to\infty}\frac{1}{2T}\int_{-(T+|a|)+a}^{(T+|a|)+a}f(x)\,dx \\ &= \lim_{T\to\infty}\frac{T+|a|}{T}\frac{1}{2(T+|a|)}\int_{-(T+|a|)+a}^{(T+|a|)+a}f(x)\,dx \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a}f(x)\,dx.\end{aligned}$$

It follows that $\lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a}f(x)\,dx = \lim\limits_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)\,dx.$ □

It will also be useful to note that there is a form of Parseval's identity that is valid on B^2:

Theorem 1.3.7 (Parseval's Identity) *For any* $f \in B^2$, $\|f\|_{B^2} = \left(\sum\limits_{n\in\mathbb{Z}}|\widehat{f}(\lambda_n)|^2\right)^{\frac{1}{2}}$.

1.3.2 Specialised Definitions and Results

This section collects some of the more specialised theory pertaining to Besicovitch almost periodic functions that will be necessary for the remainder of the first part of this thesis. The proofs provided are largely adaptations by the author of proofs of equivalent standard results on L^p.

The following result shows that convolution on $\mathbb{R}$ of almost periodic trigonometric polynomials with $L^1(\mathbb{R})$ functions behaves in an analogous way to convolution of functions in the regular Lebesgue function spaces:

Theorem 1.3.8 *If $f \in \mathscr{P}$ and $\phi \in L^1(\mathbb{R})$, then the convolution of f and ϕ on $\mathbb{R}$ is also in $\mathscr{P}$ and is equal to $\sum_{n\in\mathbb{Z}} \widehat{f}(\lambda_n)\widehat{\phi}(\lambda_n)e^{2\pi i\lambda_n \cdot}$.*

Proof This is a consequence of the following simple calculation:

$$\begin{aligned} f*\phi &= \int_{\mathbb{R}} \Big(\sum_{n\in\mathbb{Z}} \widehat{f}(\lambda_n)e^{2\pi i\lambda_n(\cdot-y)}\Big)\phi(y)\,dy \\ &= \sum_{n\in\mathbb{Z}} \widehat{f}(\lambda_n)e^{2\pi i\lambda_n\cdot}\int_{\mathbb{R}}\phi(y)e^{-2\pi i\lambda_n y}\,dy \\ &= \sum_{n\in\mathbb{Z}} \widehat{f}(\lambda_n)\widehat{\phi}(\lambda_n)e^{2\pi i\lambda_n\cdot}. \end{aligned}$$

It is noted that the interchange of the sum and integral is permitted as the sum possesses only finitely many non-zero terms since $f \in \mathscr{P}$. □

The following three operators acting on a function $f \in B^p$, $p \in [1,\infty)$ are defined for $a \in \mathbb{R}$:

$$M_a f(x) := f(x)e^{2\pi iax}, \quad \tau_a f(x) := f(x-a), \quad D_a f(x) := f(a^{-1}x).$$

The results of some elementary calculations are given now for future reference:

Proposition 1.3.9 *For any $f \in B^p$ and $\lambda, a \in \mathbb{R}$,*

$$\widehat{M_a f}(\lambda) = \widehat{f}(\lambda-a), \quad \widehat{\tau_a f}(\lambda) = e^{-2\pi i\lambda a}\widehat{f}(\lambda), \quad \widehat{D_a f}(\lambda) = \widehat{f}(a\lambda).$$

The adjoints of these operators with respect to the B^2 inner product can also easily be calculated as the following (using Theorem 1.3.6 in the case of translation):

$$M_\eta^* = M_{-\eta}, \quad \tau_y^* = \tau_{-y}, \quad D_\alpha^* = D_{\alpha^{-1}}.$$

The modulation, translation and dilation operators are all isometries on any Besicovitch space. More generally, an operator T acting on functions in B^p for some $p \in [1, \infty)$ can be said to be a bounded operator if it satisfies the usual property, namely that $\|Tf\|_{B^p} \lesssim \|f\|_{B^p}$. Here, the term "operator" may include non-linear operators in addition to linear operators, so long as they are homogeneous of degree 1. Extending this concept, there is also a notion of weak boundedness of operators that generalises the same concept on L^p:

Definition 1.3.10 (Weak boundedness on B^p) *An operator Ω acting on functions in B^p for some $p \in [1, \infty)$ can be said to be weakly bounded on B^p if*

$$\lim_{T \to \infty} \frac{1}{2T} |\{x \in [-T, T] : |\Omega f(x)| > t\}| \lesssim \left(\frac{\|f\|_{B^p}}{t} \right)^p$$

for any $t > 0$ and any $f \in B^p$.

The weak Besicovitch quasi-norm is defined as

$$\|f\|_{B^{p,\infty}} := \sup_{t>0} \left(t^p \lim_{T \to \infty} \frac{1}{2T} |\{x \in [-T, T] : |f(x)| > t\}| \right)^{\frac{1}{p}}$$

for any $p \in [1, \infty)$.

It is noted that the $B^{p,\infty}$ quasi-norm is bounded above by the B^p norm by Chebyshev's inequality. A proof of the existence of the limits above is given in Sections 4 and 5 of [69].

It will be useful to establish that this quasi-norm is equivalent to a norm, acting on trigonometric polynomials. This requires two lemmata, the first of which provides an alternative expression for the B^p norm:

Lemma 1.3.11 *For any trigonometric polynomial f and any $p \in [1, \infty)$,*

$$\|f\|_{B^p}^p = p \int_0^\infty t^{p-1} \lim_{T\to\infty} \frac{1}{2T} |\{x \in [-T, T] : |f(x)| > t\}| \, dt.$$

Proof By direct calculation, Fubini's theorem and the dominated convergence theorem, it can be seen that

$$\begin{aligned}
&\|f\|_{B^p}^p \\
= \ & \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_0^{|f(x)|} p t^{p-1} \, dt \, dx \\
= \ & p \int_0^\infty t^{p-1} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \chi_{\{x\in\mathbb{R}:|f(x)|>t\}}(x) \, dx \, dt \\
= \ & p \int_0^\infty t^{p-1} \lim_{T\to\infty} \frac{1}{2T} |\{x \in [-T, T] : |f(x)| > t\}| \, dt
\end{aligned}$$

as required. □

The following set function, μ, acting on certain measurable sets $E \subseteq \mathbb{R}$, will be notationally useful for the second lemma as well as the remainder of the first part of this thesis:

$$\mu(E) := \lim_{T\to\infty} \frac{1}{2T} |E \cap [-T, T]|.$$

It is remarked that this set function is not a (Borel) measure. Indeed, $\mu\big(\bigcup_{n\in\mathbb{Z}} [n, n+1)\big) = \mu(\mathbb{R}) = 1$, but $\mu([n, n+1)) = 0$ for all $n \in \mathbb{Z}$.

This definition given, the following can be established:

Lemma 1.3.12 *Let f be a trigonometric polynomial and choose any $p \in (1, \infty)$. Suppose that $E \subseteq \mathbb{R}$ is a measurable set such that $\mu(E)$ exists and is finite and choose any $q \in [1, p)$. Then*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{E\cap[-T,T]} |f(x)|^q \, dx \leqslant \frac{p}{p-q} \mu(E)^{1-\frac{q}{p}} \|f\|_{B^{p,\infty}}^q.$$

Proof Using Lemma 1.3.11 and making straightforward calculations, it can be seen that

$$\begin{aligned}
& \lim_{T\to\infty}\frac{1}{2T}\int_{E\cap[-T,T]}|f(x)|^q\,dx \\
= \;& q\int_0^\infty t^{q-1}\mu(\{x\in E:|f(x)|>t\})\,dt \\
\leqslant \;& q\int_0^\infty t^{q-1}\min(\mu(E),t^{-p}\|f\|_{B^{p,\infty}}^p)\,dt \\
= \;& q\left(\int_0^{\|f\|_{B^{p,\infty}}\mu(E)^{-\frac{1}{p}}} t^{q-1}\mu(E)\,dt+\int_{\|f\|_{B^{p,\infty}}\mu(E)^{-\frac{1}{p}}}^\infty t^{q-p-1}\|f\|_{B^{p,\infty}}^p\,dt\right) \\
= \;& \|f\|_{B^{p,\infty}}^q\mu(E)^{1-\frac{q}{p}}-\frac{q}{q-p}\|f\|_{B^{p,\infty}}^q\mu(E)^{1-\frac{q}{p}} \\
= \;& \frac{p}{p-q}\mu(E)^{1-\frac{q}{p}}\|f\|_{B^{p,\infty}}^q
\end{aligned}$$

as required. □

This second lemma allows the desired equivalence to be shown:

Proposition 1.3.13 *Let $p\in(1,\infty)$, and fix any $q\in(1,p)$. Then acting on trigonometric polynomials, $\|\cdot\|_{B^{p,\infty}}$ is equivalent to the norm $|||\cdot|||_{B^{p,\infty}}$ defined as follows:*

$$|||f|||_{B^{p,\infty}}:=\sup_{0<\mu(E)<\infty}\mu(E)^{\frac{1}{p}-\frac{1}{q}}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{E\cap[-T,T]}|f(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Proof Fix a trigonometric polynomial f. Since $|||f|||_{B^{p,\infty}}\leqslant\left(\frac{p}{p-q}\right)^{\frac{1}{q}}\|f\|_{B^{p,\infty}}$ follows immediately from Lemma 1.3.12, it suffices to show that $\|f\|_{B^{p,\infty}}\leqslant|||f|||_{B^{p,\infty}}$.

For any $t>0$,

$$\begin{aligned}
|||f|||_{B^{p,\infty}} &\geqslant \mu(\{x\in\mathbb{R}:|f(x)|>t\})^{\frac{1}{p}-\frac{1}{q}}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{\{x\in\mathbb{R}:|f(x)|>t\}\cap[-T,T]}|f(x)|^q\,dx\right)^{\frac{1}{q}} \\
&\geqslant \mu(\{x\in\mathbb{R}:|f(x)|>t\})^{\frac{1}{p}-\frac{1}{q}}t\mu(\{x\in\mathbb{R}:|f(x)|>t\})^{\frac{1}{q}}.
\end{aligned}$$

By uniformity in t, it follows that $\|f\|_{B^{p,\infty}}\leqslant|||f|||_{B^{p,\infty}}$. □

The following result is a natural analogue of its counterparts on $L^p(\mathbb{R})$ and $L^p(\mathbb{T})$:

Theorem 1.3.14 *Suppose that T is a linear operator that commutes with translations and is bounded on B^p for some $p \in [1,\infty)$. Then T has a multiplier representation for some multiplier $m \in L^\infty(\mathbb{R})$ in the sense that T may be defined by the property that for any $f \in B^p$, $(Tf)\hat{}(\lambda) = m(\lambda)\hat{f}(\lambda)$ for any $\lambda \in \mathbb{R}$.*

Proof Fix any $\lambda \in \mathbb{R}$. Then by translation commutation,

$$T(\tau_y e^{2\pi i\lambda\cdot}) = \tau_y T(e^{2\pi i\lambda\cdot}).$$

Considering the Fourier coefficients of each side, choose any $\gamma \in \mathbb{R}$ and observe that by linearity and the elementary calculations of Proposition 1.3.9,

$$T(\tau_y e^{2\pi i\lambda\cdot})\hat{}(\gamma) = e^{-2\pi i\lambda y} T(e^{2\pi i\lambda\cdot})\hat{}(\gamma)$$

and

$$(\tau_y T(e^{2\pi i\lambda\cdot}))\hat{}(\gamma) = e^{-2\pi i\gamma y} T(e^{2\pi i\lambda\cdot})\hat{}(\gamma).$$

As such, $e^{-2\pi i\lambda y} T(e^{2\pi i\lambda\cdot})\hat{}(\gamma) = e^{-2\pi i\gamma y} T(e^{2\pi i\lambda\cdot})\hat{}(\gamma)$ for all $y, \gamma \in \mathbb{R}$.

This implies that $T(e^{2\pi i\lambda\cdot})\hat{}(\gamma) = 0$ for any $\gamma \neq \lambda$. Further, by the boundedness property and Parseval's identity,

$$|T(e^{2\pi i\lambda\cdot})\hat{}(\lambda)| = \|T(e^{2\pi i\lambda\cdot})\|_{B^2} \leqslant \|T\|_{B^2 \to B^2}.$$

The result now follows by linearity. $\square$

1.3.3 The Bohr Compactification and Transference

The Bohr compactification of a locally compact abelian group provides means through which Besicovitch almost periodic functions can be viewed as L^p functions, at the cost of these functions having to be defined on a more complicated group than $\mathbb{R}$. Following the abstract presentation given in [125, Def. 1.1] (see also [122] and [11]), it can be defined as follows:

Definition 1.3.15 *The Bohr compactification of a locally compact abelian group G consists of a compact group G_B and a continuous homomorphism $i_B : G \to G_B$ such that for any homomorphism ϕ mapping G into a compact group Γ there exists a unique corresponding homomorphism $\phi_B : G_B \to \Gamma$ such that $\phi = \phi_B \circ i_B$.*

The term "Bohr compactification" is often used to simply refer to the group G_B; it can be seen from this definition that this group is determined uniquely up to isomorphism for each G.

An explicit expression of the Bohr compactification of a given locally compact abelian group G can be formulated in terms of its dual group, G'. Using $(G')_d$ to represent G' equipped with the discrete topology and recalling that the Pontryagin duality theorem states that the natural map from G to $(G')'$ is a topological isomorphism (see, for example, [122, Thm. 1.7.2]), the following can be seen to hold:

Proposition 1.3.16 *Given G, a locally compact abelian group, the Bohr compactification of G consists of $G_B = ((G')_d)'$ and i_B, mapping $G = (G')'$ into $((G')_d)'$, the natural embedding dual to the identity map, $\iota : (G')_d \to G'$. Further, $i_B(G)$ is dense in G_B.*

For a proof of these facts, see [125, Prop. 1.1].

The role of the Bohr compactification in the context of Besicovitch almost periodic functions is seen in the following theorem:

Theorem 1.3.17 *For any $p \in [1, \infty)$, the space of Besicovitch almost periodic functions, B^p, is isometrically isomorphic to the space $L^p(\mathbb{R}_B, \mu_{\mathbb{R}_B})$, where $\mu_{\mathbb{R}_B}$ is the Haar measure associated to $\mathbb{R}_B$.*

This is proved in [59], for example, where it is used to additionally show that for any $p \in (1,\infty)$, the dual space of B^p is $B^{p'}$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Noting that the dual spaces of $\mathbb{R}$ and $\mathbb{R}_B$ differ only in their topology, it can be seen that multiplier operators on B^p and $L^p(\mathbb{R})$ can be considered to be in direct correspondence. Further, in his 1965 paper on transference of L^p multipliers[96], de Leeuw established the following:

Theorem 1.3.18 *A continuous function $m \in L^\infty(\mathbb{R})$ is a Fourier multiplier corresponding to a bounded operator on $L^p(\mathbb{R})$ if and only if it is a Fourier multiplier corresponding to a bounded operator on $L^p(\mathbb{R}_B)$. In this case, the norms of the two operators corresponding to m coincide.*

Given the isomorphism between $L^p(\mathbb{R}_B)$ and B^p, this result allows boundedness of a range of operators on B^p to be deduced from boundedness of their counterparts on $L^p(\mathbb{R})$. Further, in 1995, Nakhlé Asmar, Earl Berkson and T. Alastair Gillespie[7] showed that this transference can be extended to the context of maximal multiplier operators, proving the following two results*:

Theorem 1.3.19 *Let $(m_j)_{j\in\mathbb{N}} \subseteq L^\infty(\mathbb{R})$ be a sequence of continuous functions that are Fourier multipliers corresponding to bounded operators $(T_j)_{j\in\mathbb{N}}$ on $L^p(\mathbb{R})$ for some $p \in [1,\infty)$. Then if the induced maximal operator on $L^p(\mathbb{R})$, defined as $\sup_{j\in\mathbb{N}} |T_j f|$, is a weakly bounded operator, so is the corresponding maximal operator on $L^p(\mathbb{R}_B)$.*

Theorem 1.3.20 *Let $(m_j)_{j\in\mathbb{N}} \subseteq L^\infty(\mathbb{R})$ be a sequence of continuous functions. Then given any $p \in [1,\infty)$, $(m_j)_{j\in\mathbb{N}}$ is a sequence of Fourier multipliers corresponding to bounded operators on $L^p(\mathbb{R})$ if and only if it is a sequence of Fourier multipliers corresponding to bounded operators on $L^p(\mathbb{R}_B)$; if this is the case, the norms of the induced maximal operators on $L^p(\mathbb{R})$ and $L^p(\mathbb{R}_B)$, defined as $\sup_{j\in\mathbb{N}} |T_j f|$, coincide.*

*The two results given here correspond to Theorems 5.1 and 6.5 in [7]. The results there are presented in the context of a general locally compact abelian group rather than for $\mathbb{R}$.

In particular, if for some $p \in [1,\infty)$, $(m_j)_{j\in\mathbb{N}}$ is a sequence of continuous functions on $\mathbb{R}$ such that the operators $(m_j \widehat{f})^{\vee}$ are bounded on $L^p(\mathbb{R})$ and the maximal operator, $\sup_{j\in\mathbb{N}} |(m_j\widehat{f})^{\vee}|$ is also weakly or strongly bounded on $L^p(\mathbb{R})$, it follows from the above theorems that the operator

$$\sup_{j\in\mathbb{N}} \Big| \sum_{n\in\mathbb{Z}} m_j(\lambda_n)\widehat{f}(\lambda_n)e^{2\pi i\lambda_n\cdot} \Big|$$

is correspondingly weakly or strongly bounded as an operator on B^p.

As far as this thesis is concerned, the multipliers of greatest interest will be the characteristic functions $m_j = \chi_{[-j,j]}$ which correspond to Fourier partial summation. Unfortunately, these multipliers do not strictly satisfy the hypotheses of the above theorems since they are not continuous. Nonetheless, all three of the above theorems can be generalised to remove the hypothesis of continuity. As a pair of corollaries to his statement of Theorem 1.3.18 in [96], de Leeuw extended his theorem to require only that m be bounded and measurable (although the assertion of equality of norms was lost in the process). He did this by defining a non-negative, continuous and compactly supported function ϕ with integral 1; this function gives rise to the approximate identity $(\phi_\varepsilon)_{\varepsilon>0}$, where $\phi_\varepsilon := \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right)$. The functions $\phi_\varepsilon * m$ are then bounded and continuous functions which approximate m and after some manipulation, the original theorem can be used to conclude the following result:

Theorem 1.3.21 *A bounded measurable function m on $\mathbb{R}$ is a Fourier multiplier corresponding to a bounded operator on $L^p(\mathbb{R})$ if and only if it is a Fourier multiplier corresponding to a bounded operator on $L^p(\mathbb{R}_B)$.*

The same technique can be applied to the theorems of Asmar, Berkson and Gillespie to remove the hypothesis of continuity from Theorems 1.3.19 and 1.3.20. This process is detailed by the authors in their earlier paper, [6], in which they generalise the key result that underpins their proof of Theorem 1.3.20 in exactly this way.* As such, the following results hold:

*Specifically, in [6], Theorem 4.1 (which is Theorem 2.1 from [7]) is generalised as Corollary 4.5 to replace continuity of the sequence of multipliers with the hypothesis that they are bounded and measurable. As in [7], the context of these theorems is the general setting of locally compact abelian groups rather than $\mathbb{R}$ and so this generalisation requires the slightly awkward technical hypothesis that a suitable approximate identity sequence exists; this is presumably why this generalisation of Theorems 1.3.19 and 1.3.20 is not explored in [7].

Theorem 1.3.22 *Let $(m_j)_{j\in\mathbb{N}}$ be a sequence of bounded measurable functions on $\mathbb{R}$ that are Fourier multipliers corresponding to bounded operators $(T_j)_{j\in\mathbb{N}}$ on $L^p(\mathbb{R})$ for some $p \in [1,\infty)$. Then if the induced maximal operator on $L^p(\mathbb{R})$, defined as $\sup_{j\in\mathbb{N}} |T_j f|$, is a weakly bounded operator, so is the corresponding maximal operator on $L^p(\mathbb{R}_B)$.*

Theorem 1.3.23 *Let $(m_j)_{j\in\mathbb{N}}$ be a sequence of bounded measurable functions on $\mathbb{R}$. Then given any $p \in [1,\infty)$, $(m_j)_{j\in\mathbb{N}}$ is a sequence of Fourier multipliers corresponding to bounded operators on $L^p(\mathbb{R})$ if and only if it is a sequence of Fourier multipliers corresponding to bounded operators on $L^p(\mathbb{R}_B)$.*

It is thus the case that this transference theory is applicable even to the discontinuous multipliers that will be considered in the remainder of this part of the present thesis.

CHAPTER 2

THE CARLESON OPERATOR AND AN APPROACH TO TIME-FREQUENCY ANALYSIS ON B^2

2.1 Introduction to the Problem

The Carleson operator acting on trigonometric polynomials $f \in \mathscr{P}$ is defined in the following way:

$$\mathscr{C} f(x) = \sup_{\xi \in \mathbb{R}} \Big| \sum_{-\infty < \lambda_n < \xi} \widehat{f}(\lambda_n) e^{2\pi i \lambda_n x} \Big|.$$

This operator is weakly bounded as an operator on B^2 functions. This can be stated as the following theorem:

Theorem 2.1.1 *The operator $\mathscr{C}$ continuously extends to the class of B^2 functions and for all such functions,*

$$\lim_{T \to \infty} \frac{1}{2T} |\{x \in [-T, T] : |\mathscr{C} f(x)| > t\}| \lesssim \left(\frac{\|f\|_{B^2}}{t} \right)^2$$

for all $t > 0$.

This result follows abstractly from Carleson's Theorem on $L^2(\mathbb{R})$ and the transference results mentioned in Section 1.3.3. Nonetheless, the motivation of this part of the present thesis is establishing almost periodic analogues of the techniques used by Lacey and Thiele in [92] to prove Carleson's Theorem. Ultimately, such techniques may be useful for proving results on B^2 in contexts where a transference framework is not as well developed, for example boundedness results for multilinear operators. As has already been discussed in Section 1.1, the heritage of Lacey and Thiele's techniques is in the problem of boundedness of the bilinear Hilbert transform and in his book, [142], Thiele suggests that time-frequency techniques of the type that Lacey and he used to prove Carleson's theorem on $L^2(\mathbb{R})$ are applicable to a class of problems that is "rather large with many interesting ramifications."[142, p. 1] It thus seems reasonable to suggest that developing a good understanding of time-frequency techniques on almost periodic functions would be valuable for a better understanding of the field as a whole.

The remainder of this part of the present thesis is devoted to an exposition of partial progress by the author towards adapting Lacey and Thiele's techniques in a way sufficient to proof Theorem 2.1.1. Whilst the proposed scheme does not yet provide a complete proof of weak B^2 boundedness of the Carleson operator, it will be seen to replicate many aspects of Lacey and Thiele's argument. That a time-frequency approach to bounding the Carleson operator on B^2 is appropriate is suggested by the fact that the almost periodic Carleson operator essentially possesses the same modulation, translation and dilation symmetries as its counterparts on $\mathbb{T}$ and $\mathbb{R}$; these will be explored later.

It should be clarified that, at present, nothing about convergence of Fourier series for functions in B^2 can be said to follow from weak B^2 boundedness of the Carleson operator. Since equivalent functions in B^2 can differ on sets of positive measure, it seems likely that any statement about convergence would need to be formulated in a more sophisticated sense than "almost everywhere".

To date, there has been little study of problems related to convergence of almost periodic Fourier series. One of the early papers in the area was by Salomon Bochner, published in

1927[18], in which conditions under which some of the early Fourier series convergence results can be generalised are considered. Most notably, various papers by Evgeniya Bredihina were published between 1960 and 1970, establishing convergence and divergence results, mostly for the Bohr class or variations of it ([28], [29], [30], [32], [33], [36], [37], [38], [39] and [40]), but also for the Stepanov classes of almost periodic functions ([31] and [34]). In addition, in [35], Bredihina established a result on absolute convergence of Fourier series in the Besicovitch space B^2.

Of Bredihina's papers, particular attention is drawn to [34] in which she establishes almost everywhere convergence of dyadic partial sums of Fourier series for functions in the Stepanov space S^2 satisfying the separation condition (Definition 1.3.5). The present author's previous thesis[11] (and subsequent paper, [10]) provides an alternative proof and extension of this result, showing boundedness of an appropriate maximal operator in the Stepanov spaces S^{2^k} for $k \in \mathbb{N}$ and proving that such boundedness does imply almost everywhere convergence.

2.2 Tile Localisation

Since the almost periodic Carleson operator will be considered from a time-frequency analytic perspective, it will be necessary to split it into pieces that are localised to tiles in the time-frequency plane. The aim, for a given function $f \in \mathscr{P}$ and a tile s, is to find a localising function $\psi_{s,f}$ such that $\langle\!\langle f, \psi_{s,f} \rangle\!\rangle \psi_{s,f}$ represents the part of f contained in s in some sense. This function will be the almost periodic analogue of the localising function ϕ_s from Section 1.2. Functions in $B^2 \backslash \mathscr{P}$ will not be considered directly in the time-frequency analytic framework – indeed, the model for the Carleson operator will be constructed in way that is not well-defined for general functions in B^2 and so all estimates pertaining to this model will apply only to the class of trigonometric polynomials. In Section 3.4, a density argument that extends results on the Carleson operator itself from $\mathscr{P}$ to the whole of B^2 will be considered.

To allow certain estimates on Besicovitch norms to proceed, the function $\psi_{s,f}$ should exhibit some amount of almost periodicity akin to f, assuring that if it is well localised to a particular time interval, it will also be, in some rough sense, well localised in an appropriate almost periodic manner to translates of that interval. To speak more concretely, if f is a periodic trigonometric polynomial with period $p \in \mathbb{R}^+$, the localising function, $\psi_{s,f}$, should also be taken to be periodic with period p so that localisation will occur on p-translates of the time interval associated with s. This notion, together with the fact that periodic functions of all different periods co-exist in B^2, explains why the localising function on B^2 should have some dependence on the function that it is localising, unlike its counterpart on $L^2(\mathbb{R})$ (or indeed any counterpart on $L^2(\mathbb{T})$). However, the dependence on f in $\psi_{s,f}$ should ideally be sufficiently mild as to allow various estimates to be made independently of f. Defining the localising function in this way also suggests that the time projection of tiles considered should be restricted to some finite interval dependent on f (the "time window").

Continuing to consider periodic trigonometric polynomials as a special case, it is noted that the period of any such f coincides with $\frac{1}{\alpha_f}$. This suggests that for general $f \in \mathscr{P}$, a selection of tiles with time projections distributed over an interval of width $\frac{1}{\alpha_f}$ might be appropriate.* In this spirit, the following collection of dyadic tiles is defined for any $\rho, \alpha \in \mathbb{R}$:

$$\mathbb{D}_{\rho,\alpha} := \{(\alpha^{-1} I_s + \rho) \times \alpha\omega_s : s \in \mathbb{D}, I_s \subseteq [0,1)\}.$$

It can be seen that the ρ parameter determines the location of the restricted "time window", whilst α determines the scale of the tiles and the size of the window and thus, given that there is a widest scale in the collection $\mathbb{D}_{\rho,\alpha}$, the "resolution" of the localisation that will be carried out.

The collection of such tiles at a single scale, $k \in \mathbb{Z}$, will be denoted by

$$\mathbb{D}_{\rho,\alpha,k} := \{s \in \mathbb{D}_{\rho,\alpha} : |I_s| = \tfrac{2^k}{\alpha}\}.$$

*For $f \in \mathscr{P}$, the constant α_f is always well-defined when f consists of more than one term. Since boundedness of the Carleson operator acting on trigonometric polynomials with fewer than two terms is trivial, such functions will not generally be considered here.

It is noted that since $I_s \subseteq [\rho, \rho + \frac{1}{\alpha})$ for any tile $s \in \mathbb{D}_{\rho,\alpha}$, necessarily $k \leqslant 0$ whenever this collection is non-empty.

The basis of the almost periodic localising function will be the $L^2(\mathbb{R})$ localising function, which was discussed in Section 1.2. As there, fix $\phi \in \mathscr{S}(\mathbb{R})$ such that $\widehat{\phi}(\xi) \in \mathbb{R}_0^+$ for any $\xi \in \mathbb{R}$ and so that $\operatorname{supp}(\widehat{\phi}) \subseteq [-\frac{1}{20}, \frac{1}{20}]$. For any tile s of area 1 in the time-frequency plane define

$$\phi_s(x) := |I_s|^{-\frac{1}{2}} \phi\left(\frac{x - c(I_s)}{|I_s|}\right) e^{2\pi i c(\omega_s^-) x}.$$

As before, it can be seen that

$$\widehat{\phi_s}(\xi) = |\omega_s|^{-\frac{1}{2}} \widehat{\phi}\left(\frac{\xi - c(\omega_s^-)}{|\omega_s|}\right) e^{2\pi i (c(\omega_s^-) - \xi) c(I_s)}.$$

This is simply the $L^2(\mathbb{R})$ localising function, albeit taking a wider collection of tiles as a parameter. As observed previously, $\widehat{\phi_s}$ is supported in $\frac{1}{5} \star \omega_s^-$ and ϕ_s is "roughly supported" in I_s.

The almost periodic localising function, $\psi_{s,f}$, for a function $f \in \mathscr{P}$ is defined as follows:

Definition 2.2.1 (Almost Periodic Localising Function) *For each function $f \in \mathscr{P}$, define the set Λ_f to be the set of all numbers in $[0, \alpha_f)$ that are congruent modulo α_f to an element of the spectrum of f, $\sigma(f)$. Then for a tile s of area 1 in the time-frequency plane, the almost periodic localising function, $\psi_{s,f}$, is defined as*

$$\psi_{s,f} := \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \Lambda_f}} \sum_{m \in \mathbb{Z}} \widehat{\phi_s}(\lambda_n + \alpha_f m) e^{2\pi i (\lambda_n + \alpha_f m) \cdot},$$

where ϕ_s is the $L^2(\mathbb{R})$ localising function.

If, for a given function $f \in \mathscr{P}$, members of $\sigma(f)$ are considered to be "equivalent" if they are congruent modulo α_f, the set Λ_f is formed of all representatives of each equivalence class under this relation that are contained in $[0, \alpha_f)$; it is essentially a base set of frequencies representing the structure of f. It should be clarified that these representatives do not have to be elements of $\sigma(f)$ themselves. For example, for the function

$$f := 1 + e^{2\pi i \cdot} + e^{2\pi^2 i \cdot},$$

the set Λ_f is equal to $\{0, \pi - 3\}$ (here, $\alpha_f = 1$ and $\sigma(f) = \{0, 1, \pi\}$). In the case of a periodic function, Λ_f is always the singleton set $\{0\}$. Note that as $f \in \mathscr{P}$, the quantity $|\Lambda_f|$ is always finite.

The spectrum of the localising function associated to a function $f \in \mathscr{P}$ is made up of the spectrum of f together with its α_f-translates. Recalling that $\widehat{\phi_s}$ is supported in a sub-interval of ω_s^-, calculating directly and using orthonormality,

$$\langle f, \psi_{s,f} \rangle = \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(f) \cap \omega_s^-}} \widehat{f}(\lambda_n) \overline{\widehat{\phi_s}(\lambda_n)}.$$

As such, in spite of the additional translated frequencies present in the definition of the localising function, this inner product provides the desired frequency localisation of f to a tile s.

Time localisation can be seen as follows:

$$\begin{aligned}
\psi_{s,f}(x) &= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \Lambda_f}} \sum_{m \in \mathbb{Z}} \widehat{\phi_s}(\lambda_n + \alpha_f m) e^{2\pi i (\lambda_n + \alpha_f m) x} \\
&= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \Lambda_f}} e^{2\pi i \lambda_n x} \sum_{m \in \mathbb{Z}} \widehat{\phi}_{I_s \times (\omega_s - \lambda_n)}(\alpha_f m) e^{2\pi i \alpha_f m x} \\
&= \frac{1}{|\Lambda_f| \sqrt{\alpha_f}} \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \Lambda_f}} e^{2\pi i \lambda_n x} \sum_{m \in \mathbb{Z}} \phi_{I_s \times (\omega_s - \lambda_n)} \left(x + \frac{m}{\alpha_f} \right)
\end{aligned}$$

by Poisson summation. This shows that $\psi_{s,f}$ is well-localised to I_s and its $\frac{1}{\alpha_f}$-translates, given the localisation properties of ϕ_s. This expression can be regarded as a weighted average of periodic localising functions with the weights determined by the equivalence classes in $\sigma(f)$, representing the global structure of the function f.

Both the earlier "frequency expression" definition of $\psi_{s,f}$ given in Definition 2.2.1 and this "time expression" will be used as definitions of $\psi_{s,f}$ where appropriate throughout the remainder of this thesis. This second definition can be stated as follows:

Definition 2.2.2 (Almost Periodic Localising Function) *For each function $f \in \mathscr{P}$ and tile s of area 1 in the time-frequency plane, the almost periodic localising function, $\psi_{s,f}$, of Definition 2.2.1 can alternatively be defined as follows:*

$$\psi_{s,f} := \frac{1}{|\Lambda_f|\sqrt{\alpha_f}} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{2\pi i\lambda_n\cdot} \sum_{m\in\mathbb{Z}} \phi_{I_s\times(\omega_s-\lambda_n)}\left(\cdot+\frac{m}{\alpha_f}\right).$$

It is noted that $\psi_{s,f}$ is sensitive to "structural changes" to the function f, regardless of the magnitude of the numerical value of such changes. For example, for a function $f \in \mathscr{P}$, if ε is chosen to be small, γ is an arbitrary complex number, $\lambda \in \mathbb{R}$ is such that $\widehat{f}(\lambda) \neq 0$ and $g \in \mathscr{P}$ is defined as $f + \gamma e^{2\pi i(\lambda+\varepsilon\alpha_f)\cdot}$, the nature of the localising functions $\psi_{s,g}$ will be significantly different to that of the $\psi_{s,f}$, regardless of the magnitude of γ. This is a reflection on the strong structural change that is effected by transforming f to g; for instance, if f were periodic with period p, g would be very far from a p-periodic function in a "structural sense" as a result of the extra frequency, even though it would remain very close to the periodic function f in terms of its values if γ were very small. Nonetheless, the inner product $\langle\!\langle g, \psi_{s,g}\rangle\!\rangle$ does afford the appropriate relative importance to the additional term with respect to its magnitude.

2.3 Estimates on the Localising Function

In this section, some estimates on $\psi_{s,f}$ that will be useful throughout the rest of this part of this thesis will be provided. As a precursor to this, an estimate on the localising function on $L^2(\mathbb{R})$ will be established and this in turn requires a simple technical lemma:

Lemma 2.3.1 *For any $\alpha, \mu, \nu, a, b \in \mathbb{R}$ and $N > 1$,*

$$\int_{\mathbb{R}} \frac{\alpha 2^{\mu}}{(1+\alpha 2^{\mu}|x-a|)^N} \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|x-b|)^N}\, dx \lesssim_N \frac{\alpha 2^{\min(\mu,\nu)}}{(1+\alpha 2^{\min(\mu,\nu)}|a-b|)^N}.$$

Proof Assume without loss of generality that $\nu \leqslant \mu$. The above inequality will be shown by considering high and low values of $\alpha 2^{\nu}|a-b|$ separately.

Case 1: $\alpha 2^{\nu}|a-b| \leqslant 1$

Note first that

$$\frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|x-b|)^N} \leqslant \alpha 2^{\nu} \leqslant \frac{\alpha 2^{\nu} 2^N}{(1+\alpha 2^{\nu}|a-b|)^N}.$$

It thus follows that

$$\begin{aligned} & \int_{\mathbb{R}} \frac{\alpha 2^{\mu}}{(1+\alpha 2^{\mu}|x-a|)^N} \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|x-b|)^N}\, dx \\ \leqslant\ & \frac{\alpha 2^{\nu} 2^N}{(1+\alpha 2^{\nu}|a-b|)^N} \int_{\mathbb{R}} \frac{\alpha 2^{\mu}}{(1+\alpha 2^{\mu}|x-a|)^N}\, dx \\ \lesssim_N\ & \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|a-b|)^N}, \end{aligned}$$

as required.

Case 2: $\alpha 2^{\nu}|a-b| \geqslant 1$

Assume that $b \geqslant a$. For $x \leqslant \frac{a+b}{2}$, it can be seen that $|x-b| \geqslant \frac{1}{2}|a-b|$, so

$$\begin{aligned} & \int_{-\infty}^{\frac{a+b}{2}} \frac{\alpha 2^{\mu}}{(1+\alpha 2^{\mu}|x-a|)^N} \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|x-b|)^N}\, dx \\ \leqslant\ & \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu-1}|a-b|)^N} \int_{\mathbb{R}} \frac{\alpha 2^{\mu}}{(1+\alpha 2^{\mu}|x-a|)^N}\, dx \\ \lesssim_N\ & \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|a-b|)^N}. \end{aligned}$$

Similarly, for $x \geqslant \frac{a+b}{2}$, it follows that $|x-a| \geqslant \frac{1}{2}|a-b|$, hence

$$\int_{\frac{a+b}{2}}^{\infty} \frac{\alpha 2^{\mu}}{(1+\alpha 2^{\mu}|x-a|)^N} \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|x-b|)^N}\, dx$$

$$\begin{aligned}
&\lesssim_N \quad \frac{\alpha 2^{\mu}}{(\alpha 2^{\mu-1}|a-b|)^N}\\
&= \quad \frac{\alpha 2^{\mu} 2^{-N(\mu-1)}}{(\alpha|a-b|)^N}\\
&= \quad \frac{2^N 2^{(\nu-\mu)(N-1)}\alpha 2^{\nu}}{(\alpha 2^{\nu}|a-b|)^N}.
\end{aligned}$$

Now, $\alpha 2^{\nu}|a-b| \geqslant 1$, so $((\alpha 2^{\nu}|a-b|)^{-1}+1)^N \leqslant 2^N$, hence

$$(\alpha 2^{\nu}|a-b|)^{-N} \leqslant \frac{2^N}{(1+\alpha 2^{\nu}|a-b|)^N}.$$

It follows that

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{\infty} \frac{\alpha 2^{\mu}}{(1+\alpha 2^{\mu}|x-a|)^N} \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|x-b|)^N}\, dx\\
\leqslant\ & \frac{4^N 2^{(\nu-\mu)(N-1)}\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|a-b|)^N}\\
\leqslant\ & 4^N \frac{\alpha 2^{\nu}}{(1+\alpha 2^{\nu}|a-b|)^N},
\end{aligned}$$

which completes the proof of the lemma when $b \geqslant a$. The above steps adapt, essentially by symmetry, to the case of $a \geqslant b$. □

With this lemma established, the aforementioned estimate on the $L^2(\mathbb{R})$ localising function can be stated and proved:

Proposition 2.3.2 *For $\alpha, \rho \in \mathbb{R}$, $s, s' \in \mathbb{D}_{\rho,\alpha}$ and $M \in \mathbb{N}$,*

$$|\langle \phi_s, \phi_{s'} \rangle_{L^2(\mathbb{R})}| \lesssim_M \frac{\min\left(\frac{|I_s|}{|I_{s'}|}, \frac{|I_{s'}|}{|I_s|}\right)^{\frac{1}{2}}}{(1+\frac{|c(I_s)-c(I_{s'})|}{\max(|I_s|,|I_{s'}|)})^M}.$$

Further, if $|I_{s'}| \leqslant |I_s|$,

$$|\langle \phi_s, \phi_{s'} \rangle_{L^2(\mathbb{R})}| \lesssim_M \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \int_{I_{s'}} \frac{|I_s|^{-1}\, dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^M}.$$

Both of these estimates generalise to the class of arbitrary translations of tiles in $\mathbb{D}_{\rho,\alpha}$.

Proof Using the fact that $\phi \in \mathscr{S}(\mathbb{R})$ and Lemma 2.3.1, for any $M \in \mathbb{N}$,

$$\begin{aligned}
& |\langle \phi_s, \phi_{s'} \rangle| \\
= \quad & \left| \int_{\mathbb{R}} |I_s|^{-\frac{1}{2}} \phi\left(\frac{x - c(I_s)}{|I_s|}\right) e^{2\pi i c(\omega_s^-)x} |I_{s'}|^{-\frac{1}{2}} \phi\left(\frac{x - c(I_{s'})}{|I_{s'}|}\right) e^{2\pi i c(\omega_{s'}^-)x}\, dx \right| \\
\lesssim_M \quad & \int_{\mathbb{R}} |I_s|^{-\frac{1}{2}} \left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^{-M} |I_{s'}|^{-\frac{1}{2}} \left(1 + \frac{|x - c(I_{s'})|}{|I_{s'}|}\right)^{-M} dx \\
\lesssim_M \quad & |I_s|^{\frac{1}{2}} |I_{s'}|^{\frac{1}{2}} \frac{\min(|I_s|^{-1}, |I_{s'}|^{-1})}{(1 + \min(|I_s|^{-1}, |I_{s'}|^{-1})|c(I_s) - c(I_{s'})|)^M} \\
= \quad & \frac{\min\left(\frac{|I_s|}{|I_{s'}|}, \frac{|I_{s'}|}{|I_s|}\right)^{\frac{1}{2}}}{(1 + \frac{|c(I_s) - c(I_{s'})|}{\max(|I_s|, |I_{s'}|)})^M}.
\end{aligned}$$

Now, under the supposition that $|I_{s'}| \leqslant |I_s|$, this reduces to

$$|\langle \phi_s, \phi_{s'} \rangle| \lesssim_M \frac{|I_s|^{-\frac{1}{2}} |I_{s'}|^{\frac{1}{2}}}{(1 + \frac{|c(I_s) - c(I_{s'})|}{|I_s|})^M}.$$

Observe that

$$\begin{aligned}
& \left(1 + \frac{|c(I_{s'}) - c(I_s)|}{|I_s|}\right)^{-M} \\
\leqslant \quad & \left(\frac{1}{2} + \frac{|c(I_{s'}) - c(I_s)|}{|I_s|} + \frac{1}{2}\frac{|I_{s'}|}{|I_s|}\right)^{-M} \\
\leqslant \quad & \left(\frac{1}{2} + \frac{|x - c(I_s)|}{|I_s|}\right)^{-M} \quad \text{for any } x \in I_{s'} \\
\lesssim_M \quad & \left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^{-M}.
\end{aligned}$$

As such,

$$\begin{aligned}
|\langle \phi_s, \phi_{s'} \rangle| \quad \lesssim_M \quad & \left(\frac{|I_{s'}|}{|I_s|}\right)^{\frac{1}{2}} \int_{I_{s'}} \frac{|I_{s'}|^{-1}}{(1 + \frac{|x - c(I_s)|}{|I_s|})^M}\, dx \\
= \quad & \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \int_{I_{s'}} \frac{|I_s|^{-1}}{(1 + \frac{|x - c(I_s)|}{|I_s|})^M}\, dx,
\end{aligned}$$

as required.

The generalisation to arbitrarily translated tiles does not require any change to the above proof. □

The following properties of $\psi_{s,f}$ can now be proved:

Proposition 2.3.3 *For any $\rho \in \mathbb{R}$, $f \in \mathscr{P}$ and $s \in \mathbb{D}_{\rho,\alpha_f}$,*

$$|\psi_{s,f}(x)| \lesssim_M \frac{1}{\sqrt{\alpha_f}} |I_s|^{-\frac{1}{2}} \sum_{m\in\mathbb{Z}} \left(1 + \frac{|x + \frac{m}{\alpha_f} - c(I_s)|}{|I_s|}\right)^{-M}$$

for any choice of $M \in \mathbb{N}$. If $s' \in \mathbb{D}_{\rho,\alpha_f}$ with $|I_s| \geqslant |I_{s'}|$, then

$$|\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| \lesssim \frac{1}{|\Lambda_f|} \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \sum_{l\in\mathbb{Z}} \int_{I_{s'}+\frac{l}{\alpha_f}} \frac{|I_s|^{-1}\, dx}{(1 + \frac{|x-c(I_s)|}{|I_s|})^{10}}.$$

Further, if $\omega_s^- \cap \omega_{s'}^- = \emptyset$, then

$$\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle = 0.$$

Proof To see the first property, observe that

$$\begin{aligned}
|\psi_{s,f}(x)| &= \frac{1}{|\Lambda_f|\sqrt{\alpha_f}} \left| \sum_{\substack{n\in\mathbb{Z} \\ \lambda_n \in \Lambda_f}} e^{2\pi i \lambda_n x} \sum_{m\in\mathbb{Z}} \phi_{I_s \times (\omega_s - \lambda_n)} \left(x + \frac{m}{\alpha_f}\right) \right| \\
&= \frac{1}{|\Lambda_f|\sqrt{\alpha_f}} \left| \sum_{\substack{n\in\mathbb{Z} \\ \lambda_n \in \Lambda_f}} e^{2\pi i \lambda_n x} \sum_{m\in\mathbb{Z}} |I_s|^{-\frac{1}{2}} \phi \left(\frac{x + \frac{m}{\alpha_f} - c(I_s)}{|I_s|}\right) e^{2\pi i (c(\omega_s^-) - \lambda_n)(x + \frac{m}{\alpha_f})} \right| \\
&\leqslant \frac{1}{\sqrt{\alpha_f}} \sum_{m\in\mathbb{Z}} |I_s|^{-\frac{1}{2}} \left| \phi \left(\frac{x + \frac{m}{\alpha_f} - c(I_s)}{|I_s|}\right) \right| \\
&\lesssim_M \frac{1}{\sqrt{\alpha_f}} |I_s|^{-\frac{1}{2}} \sum_{m\in\mathbb{Z}} \left(1 + \frac{|x + \frac{m}{\alpha_f} - c(I_s)|}{|I_s|}\right)^{-M}
\end{aligned}$$

for any $M \in \mathbb{N}$.

For the other two properties, first observe that since

$$\psi_{s,f} := \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_n\in\Lambda_f}} \sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n + \alpha_f m) e^{2\pi i(\lambda_n+\alpha_f m)\cdot},$$

it can be seen that

$$\begin{aligned}
&\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle \\
= \;& \frac{\alpha_f}{|\Lambda_f|^2} \sum_{\substack{n,m\in\mathbb{Z} \\ \lambda_n\in\Lambda_f}} \sum_{\substack{n',m'\in\mathbb{Z} \\ \lambda'_n\in\Lambda_f}} \widehat{\phi_s}(\lambda_n + \alpha_f m)\overline{\widehat{\phi_{s'}}(\lambda_{n'} + \alpha_f m')}\langle\!\langle e^{2\pi i(\lambda_n+\alpha_f m)\cdot}, e^{2\pi i(\lambda_{n'}+\alpha_f m')\cdot}\rangle\!\rangle \\
= \;& \frac{\alpha_f}{|\Lambda_f|^2} \sum_{\substack{n,m\in\mathbb{Z} \\ \lambda_n\in\Lambda_f}} \widehat{\phi_s}(\lambda_n + \alpha_f m)\overline{\widehat{\phi_{s'}}(\lambda_n + \alpha_f m)}
\end{aligned}$$

by orthonormality and the support property of $\widehat{\phi_s}$ and $\widehat{\phi_{s'}}$. This immediately gives the third property. For the second property, continuing the previous line of reasoning, it follows that

$$\begin{aligned}
&|\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| \\
= \;& \frac{\alpha_f}{|\Lambda_f|^2} \left| \sum_{\substack{n,m\in\mathbb{Z} \\ \lambda_n\in\Lambda_f}} \widehat{\phi_s}(\lambda_n + \alpha_f m)\overline{\widehat{\phi_{s'}}(\lambda_n + \alpha_f m)} \right| \\
= \;& \frac{\alpha_f}{|\Lambda_f|^2} \left| \sum_{\substack{n\in\mathbb{Z} \\ \lambda_n\in\Lambda_f}} \sum_{m\in\mathbb{Z}} \widehat{\phi}_{I_s\times(\omega_s-\lambda_n)}(\alpha_f m)\overline{\widehat{\phi}_{I_{s'}\times(\omega_{s'}-\lambda_n)}(\alpha_f m)} \right| \\
= \;& \frac{\alpha_f}{|\Lambda_f|^2} \left| \sum_{\substack{n\in\mathbb{Z} \\ \lambda_n\in\Lambda_f}} \alpha_f \int_0^{\frac{1}{\alpha_f}} \frac{1}{\alpha_f} \sum_{m\in\mathbb{Z}} \phi_{I_s\times(\omega_s-\lambda_n)}\left(x + \frac{m}{\alpha_f}\right) \frac{1}{\alpha_f} \overline{\sum_{l\in\mathbb{Z}} \phi_{I_{s'}\times(\omega_{s'}-\lambda_n)}\left(x + \frac{l}{\alpha_f}\right)} dx \right| \\
= \;& \frac{1}{|\Lambda_f|^2} \left| \sum_{\substack{n\in\mathbb{Z} \\ \lambda_n\in\Lambda_f}} \int_0^{\frac{1}{\alpha_f}} \sum_{m\in\mathbb{Z}} \phi_{I_s\times(\omega_s-\lambda_n)}\left(x + \frac{m}{\alpha_f}\right) \overline{\sum_{l\in\mathbb{Z}} \phi_{I_{s'}\times(\omega_{s'}-\lambda_n)}\left(x + \frac{l}{\alpha_f}\right)} dx \right|
\end{aligned}$$

by Poisson summation and Parseval's Theorem.

Considering just the integral,

$$\left| \int_0^{\frac{1}{\alpha_f}} \sum_{m,l\in\mathbb{Z}} \phi_{I_s\times(\omega_s-\lambda_n)}\left(x + \frac{m}{\alpha_f}\right) \overline{\phi_{I_{s'}\times(\omega_{s'}-\lambda_n)}\left(x + \frac{l}{\alpha_f}\right)} dx \right|$$

$$
\begin{aligned}
&= \left| \sum_{m,l\in\mathbb{Z}} \int_{\frac{m}{\alpha_f}}^{\frac{m}{\alpha_f}+\frac{1}{\alpha_f}} \phi_{I_s\times(\omega_s-\lambda_n)}(x)\overline{\phi_{I_{s'}\times(\omega_{s'}-\lambda_n)}\left(x+\frac{l}{\alpha_f}-\frac{m}{\alpha_f}\right)}\,dx \right| \\
&= \left| \sum_{m,l\in\mathbb{Z}} \int_{\frac{m}{\alpha_f}}^{\frac{m}{\alpha_f}+\frac{1}{\alpha_f}} \phi_{I_s\times(\omega_s-\lambda_n)}(x)\overline{\phi_{(I_{s'}-\frac{l}{\alpha_f})\times(\omega_{s'}-\lambda_n)}(x)}\,dx \right| \\
&= \left| \sum_{l\in\mathbb{Z}} \langle \phi_{I_s\times(\omega_s-\lambda_n)}, \phi_{(I_{s'}-\frac{l}{\alpha_f})\times(\omega_{s'}-\lambda_n)} \rangle_{L^2(\mathbb{R})} \right| \\
&\lesssim_M \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \sum_{l\in\mathbb{Z}} \int_{I_{s'}+\frac{l}{\alpha_f}} \frac{|I_s|^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^M}
\end{aligned}
$$

for any $M \in \mathbb{N}$ by Proposition 2.3.2.

As a consequence,

$$
\begin{aligned}
|\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| &\lesssim_M \frac{1}{|\Lambda_f|^2}|\Lambda_f| \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \sum_{l\in\mathbb{Z}} \int_{I_{s'}+\frac{l}{\alpha_f}} \frac{|I_s|^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^M} \\
&= \frac{1}{|\Lambda_f|} \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \sum_{l\in\mathbb{Z}} \int_{I_{s'}+\frac{l}{\alpha_f}} \frac{|I_s|^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^M}.
\end{aligned}
$$

This completes the proof of the estimate on $|\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle|$. $\square$

Chapter 3

Discretisation and Tile Selection

3.1 The Model Operator and its Boundedness

With the localising function for time-frequency analysis on B^2 defined in Section 2.2, a natural model operator is given by the following expression (for $\rho, \xi \in \mathbb{R}$):

$$A_{\rho,\xi} f := \sum_{s \in \mathbb{D}_{\rho,\alpha_f}} \chi_{\omega_s^+}(\xi) \langle\!\langle f, \psi_{s,f} \rangle\!\rangle \psi_{s,f}.$$

The extent to which this operator accurately models almost periodic Fourier summation will be considered by exploring some of its properties over the course of this chapter. The remainder of the present section will be devoted to showing a strong form of boundedness on B^2 of this operator when acting on trigonometric polynomials. To establish this boundedness, the following generalisation of a well-known inequality of David Hilbert and Issai Schur will be used. Its proof is given in the author's previous thesis, [11, pp. 44–47], based on the proof from [106, pp. 138–140].

Theorem 3.1.1 *Let $(\lambda_k)_{k \in \mathbb{Z}} \subseteq \mathbb{R}$ be an increasing sequence such that there exists $\alpha > 0$ so that $\lambda_{k+1} - \lambda_k > \alpha$ for all $k \in \mathbb{N}$. Then the operator T acting on ℓ^2 sequences and given by the expression*

$$(T(a_j))_k := \sum_{j \in \mathbb{Z} \setminus \{k\}} \frac{a_j}{\lambda_k - \lambda_j}$$

is bounded from ℓ^2 to ℓ^2 with constant at most $\frac{\pi}{\alpha}$.

With this given, the following can be proved:

Theorem 3.1.2 *For functions $f \in \mathscr{P}$,*

$$\|A_{\rho,\xi} f\|_{B^2} \lesssim \frac{1}{|\Lambda_f|^{\frac{3}{2}}} \|f\|_{B^2}$$

with constant uniform in $\rho \in \mathbb{R}$ and $\xi \in \mathbb{R}$.

The presence of the factor $\frac{1}{|\Lambda_f|^{\frac{3}{2}}}$ adds some "strength" to this bound compared to the same bound without this factor since $|\Lambda_f| \in \mathbb{N}$. Nonetheless, this "strength" should not be misinterpreted: the definition of $A_{\rho,\xi}$ already implicitly contains a factor of $\frac{1}{|\Lambda_f|^2}$, so the bound attained here actually represents a *loss* of a factor of $\frac{1}{|\Lambda_f|^{\frac{1}{2}}}$ rather than a gain of a factor of $\frac{1}{|\Lambda_f|^{\frac{3}{2}}}$.

Subsequent bounds in this chapter for averaged and maximal operators derived from this model operator will also contain factors of positive powers of $\frac{1}{|\Lambda_f|}$ and this will be seen to be desirable in light of the link between these operators and almost periodic Fourier summation which will be explained in Section 3.3. As above, given the domain of definition of the time-frequency model of the Carleson operator, all of these bounds will apply only within the class of trigonometric polynomials. It is ultimately the fact that the achieved powers of $\frac{1}{|\Lambda_f|}$ are not sufficiently large and thus that there is a net loss of a positive power of $\frac{1}{|\Lambda_f|}$ that will mean that the present techniques are not yet adequate to provide a full proof of Theorem 2.1.1. Subject to an improvement of these powers, Theorem 2.1.1 would be proved for the class of trigonometric polynomials and a density argument applied to the Carleson operator, as detailed in Section 3.4, would establish Theorem 2.1.1 on the whole of B^2.

Proof (Theorem 3.1.2) To begin with, boundedness of the model operator defined at a single scale will be shown. To this end, for any fixed $k \in \mathbb{Z}$, define

$$A_{\rho,\xi,k} f = \sum_{s \in \mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi) \langle f, \psi_{s,f} \rangle \psi_{s,f}.$$

Recall that it can be assumed that $k \leqslant 0$ by the time window constraint on the tiles.

By expanding, appealing to some symmetry and using the estimate on $|\langle\!\langle \psi_{s,f}, \psi_{s',f}\rangle\!\rangle|$ from Proposition 2.3.3,

$$\begin{aligned}
&\|A_{\rho,\xi,k} f\|_{B^2}^2 \\
= &\sum_{s,s' \in \mathbb{D}_{\rho,\alpha_f,k}} \langle\!\langle f, \psi_{s,f}\rangle\!\rangle \overline{\langle\!\langle f, \psi_{s',f}\rangle\!\rangle} \langle\!\langle \psi_{s,f}, \psi_{s',f}\rangle\!\rangle \chi_{\omega_s^+ \cap \omega_{s'}^+}(\xi) \\
\lesssim &\frac{1}{|\Lambda_f|} \sum_{s \in \mathbb{D}_{\rho,\alpha_f,k}} |\langle\!\langle f, \psi_{s,f}\rangle\!\rangle|^2 \sum_{s' \in \mathbb{D}_{\rho,\alpha_f,k}} \left(\sum_{l \in \mathbb{Z}} \int_{I_{s'} + \frac{l}{\alpha_f}} \frac{|I_s|^{-1}\, dx}{(1 + \frac{|x - c(I_s)|}{|I_s|})^{10}} \right) \chi_{\omega_s^+ \cap \omega_{s'}^+}(\xi).
\end{aligned}$$

Since all the tiles in second sum are at a fixed scale, the $I_{s'}$ are pairwise disjoint across the collection of all $s' \in \mathbb{D}_{\rho,\alpha_f,k}$ such that $\xi \in \omega_{s'}^+$. As such, the above expression can be bounded by

$$\begin{aligned}
&\frac{1}{|\Lambda_f|} \sum_{s \in \mathbb{D}_{\rho,\alpha_f,k}} |\langle\!\langle f, \psi_{s,f}\rangle\!\rangle|^2 \left(\sum_{l \in \mathbb{Z}} \int_{[\rho, \rho + \frac{1}{\alpha_f}) + \frac{l}{\alpha_f}} \frac{|I_s|^{-1}\, dx}{(1 + \frac{|x - c(I_s)|}{|I_s|})^{10}} \right) \chi_{\omega_s^+}(\xi) \\
\lesssim &\frac{1}{|\Lambda_f|} \sum_{s \in \mathbb{D}_{\rho,\alpha_f,k}} |\langle\!\langle f, \psi_{s,f}\rangle\!\rangle|^2 \chi_{\omega_s^+}(\xi).
\end{aligned}$$

Now, observe that by direct calculation,

$$|\langle\!\langle f, \psi_{s,f}\rangle\!\rangle|^2 = \frac{\alpha_f}{|\Lambda_f|^2} \sum_{\substack{n,n' \in \mathbb{Z} \\ \lambda_n, \lambda_{n'} \in \sigma(f) \cap \omega_s^-}} \widehat{f}(\lambda_n) \overline{\widehat{f}(\lambda_{n'}) \widehat{\phi_s}(\lambda_n)} \widehat{\phi_s}(\lambda_{n'}).$$

Further,

$$\begin{aligned}
&\overline{\widehat{\phi_s}(\lambda_n)} \widehat{\phi_s}(\lambda_{n'}) \\
= &|\omega_s|^{-1} \widehat{\phi}\left(\frac{\lambda_n - c(\omega_s^-)}{|\omega_s|} \right) \widehat{\phi}\left(\frac{\lambda_{n'} - c(\omega_s^-)}{|\omega_s|} \right) e^{2\pi i (c(\omega_s^-) - \lambda_{n'}) c(I_s)} e^{-2\pi i (c(\omega_s^-) - \lambda_n) c(I_s)} \\
= &\frac{2^k}{\alpha_f} \widehat{\phi}\left(\frac{\lambda_n - c(\omega_s^-)}{|\omega_s|} \right) \widehat{\phi}\left(\frac{\lambda_{n'} - c(\omega_s^-)}{|\omega_s|} \right) e^{2\pi i (\lambda_n - \lambda_{n'}) c(I_s)}.
\end{aligned}$$

As such, choosing ω_u to be the unique interval of size $\alpha_f 2^{-k}$ such that $\xi \in \omega_u^+$,

$$
\begin{aligned}
&\|A_{\rho,\xi,k} f\|_{B^2}^2 \\
\lesssim\ & \frac{2^k}{|\Lambda_f|^3} \sum_{\substack{n,n' \in \mathbb{Z} \\ \lambda_n, \lambda_{n'} \in \sigma(f) \cap \omega_u^-}} \widehat{f}(\lambda_n) \overline{\widehat{f}(\lambda_{n'})} \widehat{\phi}\left(\frac{\lambda_n - c(\omega_u^-)}{|\omega_u|} \right) \widehat{\phi}\left(\frac{\lambda_{n'} - c(\omega_u^-)}{|\omega_u|} \right) \\
& \times \sum_{\substack{s \in \mathbb{D}_{\rho,\alpha_f,k} \\ \omega_s = \omega_u}} e^{2\pi i (\lambda_n - \lambda_{n'}) c(I_s)} \\
\lesssim\ & \frac{2^k}{|\Lambda_f|^3} \left(\sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(f) \cap \omega_u^-}} |\widehat{f}(\lambda_n)|^2 \right)^{\frac{1}{2}} \\
& \times \left(\sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(f) \cap \omega_u^-}} \left| \sum_{\substack{n' \in \mathbb{Z} \\ \lambda_{n'} \in \sigma(f) \cap \omega_u^-}} \overline{\widehat{f}(\lambda_{n'})} \widehat{\phi}\left(\frac{\lambda_{n'} - c(\omega_u^-)}{|\omega_u|} \right) \sum_{\substack{s \in \mathbb{D}_{\rho,\alpha_f,k} \\ \omega_s = \omega_u}} e^{2\pi i (\lambda_n - \lambda_{n'}) c(I_s)} \right|^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

by the Cauchy–Schwarz inequality.

Considering the positions of the tiles involved, it can be seen that

$$
\sum_{\substack{s \in \mathbb{D}_{\rho,\alpha_f,k} \\ \omega_s = \omega_u}} e^{2\pi i (\lambda_n - \lambda_{n'}) c(I_s)} = \sum_{l=0}^{2^{-k}-1} e^{2\pi i (\lambda_n - \lambda_{n'}) (\rho + \frac{l 2^k}{\alpha_f} + \frac{2^{k-1}}{\alpha_f})},
$$

so by summing a geometric series,

$$
\begin{aligned}
&\|A_{\rho,\xi,k} f\|_{B^2}^2 \\
\lesssim\ & \frac{2^k}{|\Lambda_f|^3} \left(\sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(f) \cap \omega_u^-}} |\widehat{f}(\lambda_n)|^2 \right)^{\frac{1}{2}} \left[\left(\sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(f) \cap \omega_u^-}} \left| \sum_{\substack{n' \in \mathbb{Z} \setminus \{n\} \\ \lambda_{n'} \in \sigma(f) \cap \omega_u^-}} \overline{\widehat{f}(\lambda_{n'})} \widehat{\phi}\left(\frac{\lambda_{n'} - c(\omega_u^-)}{|\omega_u|} \right) \right. \right. \right. \\
& \left. \left. \left. \times e^{-2\pi i \lambda_{n'} (\rho + \frac{2^{k-1}}{\alpha_f})} \frac{1 - e^{2\pi i (\lambda_n - \lambda_{n'}) \alpha_f^{-1}}}{1 - e^{2\pi i (\lambda_n - \lambda_{n'}) 2^k \alpha_f^{-1}}} \right|^2 \right)^{\frac{1}{2}} + 2^{-k} \left(\sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(f) \cap \omega_u^-}} |\widehat{f}(\lambda_n)|^2 \right)^{\frac{1}{2}} \right].
\end{aligned}
$$

Now, for each scale k, define the set

$$\Omega_k := \left\{ j \in \mathbb{Z} : \widehat{\phi}\left(\frac{\lambda_j - c(\omega_u^-)}{|\omega_u|}\right) \neq 0 \right\}.^*$$

Also, define the operator T_k acting on sequences (a_j) such that $\operatorname{supp}(a_j) \subseteq \Omega_k$ as

$$(T_k(a_j))_l = \begin{cases} \displaystyle\sum_{j \in \mathbb{Z}\setminus\{l\}} a_j \left(\frac{1 - e^{2\pi i(\lambda_l - \lambda_j)\alpha_f^{-1}}}{1 - e^{2\pi i(\lambda_l - \lambda_j)2^k \alpha_f^{-1}}}\right), & l \in \Omega_k, \\ 0, & l \notin \Omega_k. \end{cases}$$

Suppose that this operator continuously extends to ℓ^2 with

$$\|(T_k(a_j)_l)\|_{\ell^2} \lesssim 2^{-k} \|(a_j)\|_{\ell^2}.$$

Then it follows that

$$\|A_{\rho,\xi,k} f\|_{B^2} \lesssim \frac{1}{|\Lambda_f|^{\frac{3}{2}}} \Bigg(\sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(f) \cap \omega_u^-}} |\widehat{f}(\lambda_n)|^2 \Bigg)^{\frac{1}{2}}.$$

Combining the different scales,

$$\begin{aligned} & \|A_{\rho,\xi} f\|_{B^2}^2 \\ = & \left\| \sum_{k=-\infty}^{0} A_{\rho,\xi,k} f \right\|_{B^2}^2 \\ = & \sum_{k,k'=-\infty}^{0} \sum_{s \in \mathbb{D}_{\rho,\alpha_f,k}} \sum_{s' \in \mathbb{D}_{\rho,\alpha_f,k'}} \langle f, \psi_{s,f} \rangle \overline{\langle f, \psi_{s',f} \rangle} \langle \psi_{s,f}, \psi_{s',f} \rangle \chi_{\omega_s^+ \cap \omega_{s'}^+}(\xi). \end{aligned}$$

For fixed ξ, it can be seen that any two tiles, s and s' of different sizes with $\xi \in \omega_s^+ \cap \omega_{s'}^+$ satisfy $\omega_s^- \cap \omega_{s'}^- = \emptyset$, and hence are also such that $\langle \psi_{s,f}, \psi_{s',f} \rangle = 0$ (by Proposition 2.3.3). Consequently, as the choices of ω_u^- are disjoint for different scales, it follows that the estimates on

*The reader is reminded the the choice of ω_u is implicitly dependent on k.

the individual scales may be combined:

$$\begin{aligned}
&\|A_{\rho,\xi}f\|_{B^2}^2\\
= &\sum_{k=-\infty}^{0}\sum_{s,s'\in\mathbb{D}_{\rho,\alpha_f,k}}\chi_{\omega_s^+}(\xi)\chi_{\omega_{s'}^+}(\xi)\langle f,\psi_{s,f}\rangle\overline{\langle f,\psi_{s',f}\rangle}\langle\psi_{s,f},\psi_{s',f}\rangle\\
= &\sum_{k=-\infty}^{0}\|A_{\rho,\xi,k}f\|_{B^2}^2\\
\lesssim &\frac{1}{|\Lambda_f|^3}\sum_{n\in\mathbb{Z}}|\widehat{f}(\lambda_n)|^2\\
= &\frac{1}{|\Lambda_f|^3}\|f\|_{B^2}^2 \text{ by Parseval's Theorem.}
\end{aligned}$$

It thus remains only to prove that for sequences (a_j) that are compactly supported in Ω_k, $\|(T_k(a_j)_l)\|_{\ell^2}\lesssim 2^{-k}\|(a_j)\|_{\ell^2}$ as claimed. To prove this, note first that

$$\|(T_k(a_j)_l)\|_{\ell^2}\leqslant\left(\sum_{l\in\Omega_k}\left|\sum_{j\in\Omega_k\setminus\{l\}}\frac{a_j}{1-e^{2\pi i(\lambda_l-\lambda_j)2^k\alpha_f^{-1}}}\right|^2\right)^{\frac{1}{2}}+\left(\sum_{l\in\Omega_k}\left|\sum_{j\in\Omega_k\setminus\{l\}}\frac{\widetilde{a}_j}{1-e^{2\pi i(\lambda_l-\lambda_j)2^k\alpha_f^{-1}}}\right|^2\right)^{\frac{1}{2}}$$

where $\widetilde{a}_j=a_je^{-2\pi i\lambda_j\alpha_f^{-1}}$.

As $\|(\widetilde{a}_j)\|_{\ell^2}=\|(a_j)\|_{\ell^2}$, it suffices to consider just the first term. Further, in a similar way, the first term may be replaced by

$$\begin{aligned}
&\left(\sum_{l\in\Omega_k}\left|\sum_{j\in\Omega_k\setminus\{l\}}\frac{a_j}{\sin(\pi(\lambda_l-\lambda_j)2^k\alpha_f^{-1})}\right|^2\right)^{\frac{1}{2}}\\
\leqslant &\left(\sum_{l\in\Omega_k}\left|\sum_{j\in\Omega_k\setminus\{l\}}a_j\left(\frac{1}{\sin(\pi(\lambda_l-\lambda_j)2^k\alpha_f^{-1})}-\frac{1}{\pi(\lambda_l-\lambda_j)2^k\alpha_f^{-1}}\right)\right|^2\right)^{\frac{1}{2}}\\
&+\frac{\alpha_f}{\pi 2^k}\left(\sum_{l\in\Omega_k}\left|\sum_{j\in\Omega_k\setminus\{l\}}\frac{a_j}{\lambda_l-\lambda_j}\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By Theorem 3.1.1, the second term of this expression is bounded by a constant multiple of

$2^{-k}\|(a_j)\|_{\ell^2}$, so it only remains to consider the first term. The desired bound is

$$\Big\|\big(\sum_{j\in\Omega_k\setminus\{l\}} a_j K_k(j,l)\big)_l\Big\|_{\ell^2} \lesssim 2^{-k}\|(a_j)\|_{\ell^2}$$

for the kernel

$$K_k(j,l) \coloneqq \operatorname{cosec}(\pi(\lambda_l-\lambda_j)2^k\alpha_f^{-1}) - \frac{1}{\pi(\lambda_l-\lambda_j)2^k\alpha_f^{-1}}.$$

For such an operator,

$$\begin{aligned}
&\Big\|\sum_{j\in\Omega_k\setminus\{l\}} a_j K_k(j,l)\Big\|_{\ell^1(\Omega_k)}\\
=\;& \sum_{l\in\Omega_k}\Big|\sum_{j\in\Omega_k\setminus\{l\}} a_j K_k(j,l)\Big|\\
\leqslant\;& \|(a_j)\|_{\ell^1}\sup_{j\in\Omega_k}\sum_{l\in\Omega_k\setminus\{j\}}|K_k(j,l)|
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\sum_{j\in\Omega_k\setminus\{l\}} a_j K_k(j,l)\Big\|_{\ell^\infty(\Omega_k)}\\
=\;& \sup_{l\in\Omega_k}\Big|\sum_{j\in\Omega_k\setminus\{l\}} a_j K_k(j,l)\Big|\\
\leqslant\;& \|(a_j)\|_{\ell^\infty}\sup_{l\in\Omega_k}\sum_{j\in\Omega_k\setminus\{l\}}|K_k(j,l)|.
\end{aligned}$$

As $|K_k(j,l)| = |K_k(l,j)|$, by interpolation, it suffices to show that $\sup_{l\in\Omega_k}\sum_{j\in\Omega_k\setminus\{l\}}|K_k(j,l)| \lesssim 2^{-k}$.

By Taylor expansion, for $|x|<\pi$,

$$\operatorname{cosec}(x) - \frac{1}{x} = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}2(2^{2m-1}-1)B_{2m}x^{2m-1}}{(2m)!}$$

where (B_n) is the sequence of Bernoulli numbers determined by the generating function $\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}$ for $|t|<2\pi$.[2, pp. 75, 804]

As the sequence (λ_k) is α_f separated and entirely contained in $\frac{1}{5}\star\omega_u^-$ for some ω_u of size $\alpha_f 2^{-k}$, it is certainly the case that $|\lambda_l-\lambda_j|\in[\alpha_f, 2^{-k-2}\alpha_f]$. Noting also that the Bernoulli numbers satisfy the property $(-1)^{m+1}B_{2m}>0$ for any $m\in\mathbb{N}$[2, p. 805],

$$\begin{aligned}
&\sup_{l\in\Omega_k}\sum_{j\in\Omega_k\setminus\{l\}}|K_k(j,l)| \\
\leqslant\ & \sum_{n=1}^{\frac{2^{-k}}{4}}\left|\operatorname{cosec}(\pi(n\alpha_f)2^k\alpha_f^{-1})-\frac{1}{\pi(n\alpha_f)2^k\alpha_f^{-1}}\right| \\
=\ & \sum_{n=1}^{\frac{2^{-k}}{4}}\operatorname{cosec}(\pi n2^k)-\frac{1}{\pi n2^k} \\
=\ & \sum_{n=1}^{\frac{2^{-k}}{4}}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}2(2^{2m-1}-1)B_{2m}(\pi n2^k)^{2m-1}}{(2m)!} \\
\leqslant\ & \frac{2^{-k}}{4}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}2(2^{2m-1}-1)B_{2m}(\pi\frac{2^{-k}}{4}2^k)^{2m-1}}{(2m)!} \\
\leqslant\ & \frac{2^{-k}}{4}\operatorname{cosec}\left(\frac{\pi}{4}\right) \\
=\ & \frac{\sqrt{2}}{4}2^{-k}.
\end{aligned}$$

This gives the desired estimate on the kernel and completes the proof of the boundedness of the model operator.

It is remarked that for the purpose of establishing that $\sup_{l\in\Omega_k}\sum_{j\in\Omega_k\setminus\{l\}}|K_k(j,l)|\lesssim 2^{-k}$, a less precise bound would suffice in place of the exact Taylor expansion that is used above. □

3.2 Model Operator Symmetries

As discussed in Section 1.2, the $L^2(\mathbb{R})$ maximal model operator, when restricted to a single scale, commutes with translations and modulations at the same scale and, across all scales, it commutes with dyadic dilations. These properties are a consequence of the symmetries of the $L^2(\mathbb{R})$ model operator. The present section will consider the symmetry properties of the

B^2 model operator, $A_{\rho,\xi}f := \sum_{s\in\mathbb{D}_{\rho,\alpha_f}} \chi_{\omega_s^+}(\xi)\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}$, in order to explore its relation with the almost periodic Fourier summation operator.

Firstly, recall that the symbols M, τ and D are used throughout this thesis to represent the modulation, translation and dilation operations,

$$M_a f(x) := f(x)e^{2\pi iax}, \quad \tau_a f(x) := f(x-a), \quad D_a f(x) := f(a^{-1}x),$$

and that the model operator at a single scale, k, is defined as

$$A_{\rho,\xi,k}f := \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi)\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}.$$

With respect to modulation, this operator behaves completely analogously to the equivalent operator on $L^2(\mathbb{R})$ in the sense that for any $l,k\in\mathbb{Z}$, the following identity is true:

$$M_{-l\alpha_f 2^{-k}}A_{\rho,\xi+l\alpha_f 2^{-k},k}M_{l\alpha_f 2^{-k}}f = A_{\rho,\xi,k}f.$$

To prove this, note firstly that $\Lambda_{M_{l\alpha_f 2^{-k}}f} = \Lambda_f$. Using that

$$\psi_{s,f} := \frac{\sqrt{\alpha_f}}{|\Lambda_f|}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}}\sum_{m\in\mathbb{Z}}\widehat{\phi_s}(\lambda_n+\alpha_f m)e^{2\pi i(\lambda_n+\alpha_f m)\cdot},$$

it follows that

$$\begin{aligned}
&\langle\!\langle M_{l\alpha_f 2^{-k}}f,\psi_{s,M_{l\alpha_f 2^{-k}}f}\rangle\!\rangle\\
= &\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{2\pi il\alpha_f 2^{-k}x}f(x)\frac{\sqrt{\alpha_f}}{|\Lambda_{M_{l\alpha_f 2^{-k}}f}|}\\
&\quad\times\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_{M_{l\alpha_f 2^{-k}}f}}}\sum_{m\in\mathbb{Z}}\overline{\widehat{\phi_s}(\lambda_n+\alpha_f m)}e^{-2\pi i(\lambda_n+\alpha_f m)x}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} \sum_{m\in\mathbb{Z}} \overline{\widehat{\phi_s}(\lambda_n+\alpha_f m)} \\
&\quad \times \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x) e^{-2\pi i(\lambda_n - l\alpha_f 2^{-k}+\alpha_f m)x}\,dx \\
&= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in(\sigma_f+l\alpha_f 2^{-k})\cap\omega_s^-}} \widehat{f}(\lambda_n - l\alpha_f 2^{-k})\overline{\widehat{\phi_s}(\lambda_n)} \\
&= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma_f\cap(\omega_s^- - l\alpha_f 2^{-k})}} \widehat{f}(\lambda_n)\overline{\widehat{\phi_s}(\lambda_n + l\alpha_f 2^{-k})} \\
&= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma_f\cap(\omega_s^- - l\alpha_f 2^{-k})}} \widehat{f}(\lambda_n)\overline{\widehat{\phi}_{I_s\times(\omega_s - l\alpha_f 2^{-k})}(\lambda_n)} \\
&= \langle\!\langle f, \psi_{I_s\times(\omega_s - l\alpha_f 2^{-k}),f}\rangle\!\rangle.
\end{aligned}
$$

Also,

$$
\begin{aligned}
& M_{-l\alpha_f 2^{-k}}\psi_{s,M_{l\alpha_f 2^{-k}}f} \\
&= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} \sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n+\alpha_f m) e^{2\pi i(\lambda_n - l\alpha_f 2^{-k}+\alpha_f m)\cdot} \\
&= \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} \sum_{m\in\mathbb{Z}} \widehat{\phi}_{I_s\times(\omega_s - l\alpha_f 2^{-k})}(\lambda_n+\alpha_f m) e^{2\pi i(\lambda_n+\alpha_f m)\cdot} \\
&= \psi_{I_s\times(\omega_s - l\alpha_f 2^{-k}),f}.
\end{aligned}
$$

Using these two facts,

$$
\begin{aligned}
& M_{-l\alpha_f 2^{-k}} A_{\rho,\xi+l\alpha_f 2^{-k},k} M_{l\alpha_f 2^{-k}} f \\
&= \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi + l\alpha_f 2^{-k}) \langle\!\langle M_{l\alpha_f 2^{-k}} f, \psi_{s,M_{l\alpha_f 2^{-k}}f}\rangle\!\rangle M_{-l\alpha_f 2^{-k}}\psi_{s,M_{l\alpha_f 2^{-k}}f} \\
&= \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+ - l\alpha_f 2^{-k}}(\xi) \langle\!\langle f, \psi_{I_s\times(\omega_s - l\alpha_f 2^{-k}),f}\rangle\!\rangle \psi_{I_s\times(\omega_s - l\alpha_f 2^{-k}),f} \\
&= \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi) \langle\!\langle f, \psi_{s,f}\rangle\!\rangle \psi_{s,f}
\end{aligned}
$$

$$= \quad A_{\rho,\xi,k} f.$$

This shows that the claimed modulation symmetry holds.

An appropriate symmetry for translation, analogous to that that holds for the $L^2(\mathbb{R})$ model operator, would be that for $l \in \mathbb{Z}$,

$$\tau_{-l\alpha_f^{-1}2^k} A_{\rho+l\alpha_f^{-1}2^k,\xi,k} \tau_{l\alpha_f^{-1}2^k} f = A_{\rho,\xi,k} f.$$

However, this can be seen to be false. Observe that, using the expression

$$\psi_{s,f}(x) = \frac{1}{|\Lambda_f|\sqrt{\alpha_f}} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{2\pi i\lambda_n x} \sum_{m\in\mathbb{Z}} \phi_{I_s\times(\omega_s-\lambda_n)}\left(x+\frac{m}{\alpha_f}\right),$$

it can be seen that $A_{\rho,\xi,k} f$ is equal to

$$\sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\frac{1}{|\Lambda_f|\sqrt{\alpha_f}} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{-2\pi i\lambda_n t} \overline{\sum_{m\in\mathbb{Z}} \phi_{I_s\times(\omega_s-\lambda_n)}\left(t+\tfrac{m}{\alpha_f}\right)}\, dt\right)$$
$$\times \frac{1}{|\Lambda_f|\sqrt{\alpha_f}} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{2\pi i\lambda_n x} \sum_{m\in\mathbb{Z}} \phi_{I_s\times(\omega_s-\lambda_n)}\left(x+\tfrac{m}{\alpha_f}\right).$$

However, making similar (but simpler) calculations to before, the quantity

$$\tau_{-l\alpha_f^{-1}2^k} A_{\rho+l\alpha_f^{-1}2^k,\xi,k} \tau_{l\alpha_f^{-1}2^k} f$$

is found to equal

$$\sum_{s\in\mathbb{D}_{\rho+l\alpha_f^{-1}2^k,\alpha_f,k}} \chi_{\omega_s^+}(\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\frac{1}{|\Lambda_f|\sqrt{\alpha_f}} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{-2\pi i\lambda_n t} e^{-2\pi i\lambda_n l\alpha_f^{-1}2^k}\right.$$
$$\left.\times \overline{\sum_{m\in\mathbb{Z}} \phi_{(I_s-l\alpha_f^{-1}2^k)\times(\omega_s-\lambda_n)}\left(t+\tfrac{m}{\alpha_f}\right)}\, dt\right) e^{-2\pi i c(\omega_s^-)l\alpha_f^{-1}2^k} \frac{1}{|\Lambda_f|\sqrt{\alpha_f}}$$
$$\times \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{2\pi i\lambda_n x} e^{2\pi i\lambda_n l\alpha_f^{-1}2^k} \sum_{m\in\mathbb{Z}} \phi_{(I_s-l\alpha_f^{-1}2^k)\times(\omega_s-\lambda_n)}\left(x+\tfrac{m}{\alpha_f}\right) e^{2\pi i c(\omega_s^-)l\alpha_f^{-1}2^k}.$$

This can be simplified to the following expression:

$$\sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\frac{1}{|\Lambda_f|\sqrt{\alpha_f}}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{-2\pi i\lambda_n t}e^{-2\pi i\lambda_n l\alpha_f^{-1}2^k}\right.$$
$$\left.\times\sum_{m\in\mathbb{Z}}\overline{\phi_{I_s\times(\omega_s-\lambda_n)}\left(t+\tfrac{m}{\alpha_f}\right)}\,dt\right)\frac{1}{|\Lambda_f|\sqrt{\alpha_f}}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{2\pi i\lambda_n x}e^{2\pi i\lambda_n l\alpha_f^{-1}2^k}$$
$$\times\sum_{m\in\mathbb{Z}}\phi_{I_s\times(\omega_s-\lambda_n)}\left(x+\tfrac{m}{\alpha_f}\right).$$

It can thus be seen that the additional exponential terms in the two sums in n prevent the desired symmetry from holding here.

Finally, the B^2 model operator can be seen to possess a dilation symmetry that is better than the analogous symmetry for the $L^2(\mathbb{R})$ model operator. Note first that for $y,z\in\mathbb{R}$,

$$\begin{aligned}
\phi_s(yz) &= |I_s|^{-\frac{1}{2}}\phi\left(\frac{yz-c(I_s)}{|I_s|}\right)e^{2\pi i c(\omega_s^-)yz}\\
&= y^{-\frac{1}{2}}(y^{-1}|I_s|)^{-\frac{1}{2}}\phi\left(\frac{z-y^{-1}c(I_s)}{y^{-1}|I_s|}\right)e^{2\pi i y c(\omega_s^-)z}\\
&= y^{-\frac{1}{2}}\phi_{y^{-1}I_s\times y\omega_s}(z).
\end{aligned}$$

Observe also that $\alpha_{D_y f}=y^{-1}\alpha_f$ and $\Lambda_{D_y f}=y^{-1}\Lambda_f$. These three facts given, it can be seen that for $x,y\in\mathbb{R}$,

$$\begin{aligned}
&\psi_{s,D_y f}(yx)\\
&= \frac{1}{|\Lambda_{D_y f}|\sqrt{\alpha_{D_y f}}}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_{D_y f}}} e^{2\pi i\lambda_n yx}\sum_{m\in\mathbb{Z}}\phi_{I_s\times(\omega_s-\lambda_n)}(yx+\tfrac{m}{\alpha_{D_y f}})\\
&= y^{\frac{1}{2}}\frac{1}{|\Lambda_f|\sqrt{\alpha_f}}\sum_{\substack{n\in\mathbb{Z}\\ y\lambda_n\in\Lambda_f}} e^{2\pi i(y\lambda_n)x}\sum_{m\in\mathbb{Z}}\phi_{I_s\times(\omega_s-\lambda_n)}(y(x+\tfrac{m}{\alpha_f}))\\
&= y^{\frac{1}{2}}\frac{1}{|\Lambda_f|\sqrt{\alpha_f}}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{2\pi i\lambda_n x}\sum_{m\in\mathbb{Z}}y^{-\frac{1}{2}}\phi_{y^{-1}I_s\times y\omega_s-\lambda_n}(x+\tfrac{m}{\alpha_f})\\
&= \psi_{y^{-1}I_s\times y\omega_s,f}(x).
\end{aligned}$$

Using this, for any $\rho,\xi\in\mathbb{R}$,

$$
\begin{aligned}
& D_{y^{-1}}A_{y\rho,y^{-1}\xi}D_y f \\
= & \sum_{s\in\mathbb{D}_{y\rho,\alpha_{D_y f}}} \chi_{\omega_s^+}(y^{-1}\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(y^{-1}t)\overline{\psi_{s,D_y f}(t)}\,dt\right) D_{y^{-1}}\psi_{s,D_y f} \\
= & \sum_{s\in\mathbb{D}_{y\rho,y^{-1}\alpha_f}} \chi_{y\omega_s^+}(\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\overline{\psi_{s,D_y f}(yt)}\,dt\right) \psi_{s,D_y f}(y\cdot) \\
= & \sum_{s\in\mathbb{D}_{y\rho,y^{-1}\alpha_f}} \chi_{y\omega_s^+}(\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\overline{\psi_{y^{-1}I_s\times y\omega_s,f}(t)}\,dt\right) \psi_{y^{-1}I_s\times y\omega_s,f} \\
= & \sum_{s\in\mathbb{D}_{\rho,\alpha_f}} \chi_{\omega_s^+}(\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\overline{\psi_{s,f}(t)}\,dt\right) \psi_{s,f} \\
= & A_{\rho,\xi}f.
\end{aligned}
$$

This shows that this model operator commutes perfectly with dilations.

3.3 The Averaged Operator

3.3.1 Definition

To allow for modelling of the almost periodic Carleson operator, it is necessary to generate an operator with symmetries that are more exact than the symmetries that the model operator possesses. As is the case in Lacey and Thiele's proof of boundedness of the Carleson operator on $L^2(\mathbb{R})$, this is achieved by a process of averaging. Averaging conjugations of the model operator by translation and modulation will serve to provide perfect symmetries with respect to these operations, ensuring that the model for the Carleson operator shares its symmetries. Averaging conjugations by dilation is not necessary to enforce a dilation symmetry here; indeed, as was seen in the previous section, the model operator already commutes with dilations perfectly, as the associated tiles rescale in tandem with the input function, unlike the tiles associated to the model operator on $L^2(\mathbb{R})$. However, conjugation by dilation in the $L^2(\mathbb{R})$ setting does serve to "blur" the different scales of the tiles associated to the model operator and a similar effect will be necessary here. Consequently, the model operator will also be averaged over certain rescalings; the discreteness of the scales in the model operator otherwise lead to an insufficiently precise approximation of the Carleson operator.

For $z \in [1,2]$ and ρ, $\xi \in \mathbb{R}$, the model operator with adjusted scale is defined as follows:

$$A^z_{\rho,\xi} f := \sum_{s \in \mathbb{D}_{\rho, z\alpha_f}} \chi_{\omega_s^+}(\xi) \langle f, \psi_{s,f} \rangle \psi_{s,f}.$$

Proceeding exactly as in Theorem 3.1.2, the following can be proved:

Theorem 3.3.1 *For functions $f \in \mathscr{P}$,*

$$\|A^{2^z}_{\rho,\xi} f\|_{B^2} \lesssim \frac{1}{|\Lambda_f|^{\frac{3}{2}}} \|f\|_{B^2}$$

with constant uniform in $\rho \in \mathbb{R}$, $\xi \in \mathbb{R}$ and $z \in [0,1]$.

The averaged operator, acting on functions $f \in \mathscr{P}$, is defined as follows:

$$\Pi_{\rho,\xi} f := \lim_{J,K \to \infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 M_{-\eta} \tau_{-y} A^{2^z}_{\rho,\xi+\eta} \tau_y M_\eta f \, dz \, dy \, d\eta.$$

It is necessary to show that the limiting averages in y and η exist to ensure that $\Pi_{\rho,\xi}$ is well-defined. It will suffice to consider each average applied to $A_{\rho,\xi} f$ separately for any fixed ρ, $\xi \in \mathbb{R}$ and $f \in \mathscr{P}$; the existence of the averages applied to $A^{2^z}_{\rho,\xi} f$ for each $z \in [0,1]$, and further their existence in the definition of $\Pi_{\rho,\xi}$, will follow analogously. It will first be shown that the existence of these averages will follow from the existence of averages applied to the model operator at each individual scale, $A_{\rho,\xi,k}$, for each k. To prove this, it suffices to show that for any $f \in \mathscr{P}$, there are only finitely many choices for k such that $A_{\rho,\xi,k} f$ is not identically zero (where the number of choices may depend on f).

Fix $y, \eta \in \mathbb{R}$ and note that for $k \in \mathbb{Z}$,

$$M_{-\eta}\tau_{-y}A_{\rho,\xi+\eta,k}\tau_y M_\eta f = \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi+\eta)\langle\!\langle \tau_y M_\eta f, \psi_{s,\tau_y M_\eta f}\rangle\!\rangle M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f}.$$

Assume that k is chosen such that $\alpha_f 2^{-k} \geqslant 40|\xi|$ (recall that because of the time window restriction, necessarily $k \leqslant 0$) and consider the inner product from this expression:

$$\begin{aligned}
& |\langle\!\langle \tau_y M_\eta f, \psi_{s,\tau_y M_\eta f}\rangle\!\rangle| \\
= \; & |\langle\!\langle f, M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f}\rangle\!\rangle| \\
= \; & \frac{\sqrt{\alpha_f}}{|\Lambda_f|}\Big|\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma(f)\cap(\omega_s^- -\eta)}} \widehat{f}(\lambda_n)e^{-2\pi i(\lambda_n+\eta)y}\overline{\widehat{\phi_s}(\lambda_n+\eta)}\Big| \\
\leqslant \; & \sqrt{\alpha_f}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma(f)\cap(\omega_s^- -\eta)}} |\widehat{f}(\lambda_n)||\widehat{\phi_s}(\lambda_n+\eta)|.
\end{aligned}$$

Now, as $\xi+\eta \in \omega_s^+$ and $\alpha_f 2^{-k} \geqslant 40|\xi|$,

$$\begin{aligned}
\eta & \in [c(\omega_s)-\xi, c(\omega_s)+\tfrac{1}{2}\alpha_f 2^{-k}-\xi] \\
& \subseteq [c(\omega_s)-\tfrac{1}{40}\alpha_f 2^{-k}, c(\omega_s)+\tfrac{21}{40}\alpha_f 2^{-k}] \\
& = [c(\omega_s^-)+\tfrac{9}{40}\alpha_f 2^{-k}, c(\omega_s^-)+\tfrac{31}{40}\alpha_f 2^{-k}].
\end{aligned}$$

Observe that as $\operatorname{supp}(\widehat{\phi_s}) \subseteq \frac{1}{5}\star\omega_s^- = [c(\omega_s^-)-\frac{1}{20}\alpha_f 2^{-k}, c(\omega_s^-)+\frac{1}{20}\alpha_f 2^{-k}]$, it is necessarily the case that $\widehat{\phi_s}(\lambda_n+\eta) = 0$ for $\lambda_n > c(\omega_s^-)+\frac{1}{20}\alpha_f 2^{-k}-\eta$ or $\lambda_n < c(\omega_s^-)-\frac{1}{20}\alpha_f 2^{-k}-\eta$. Further,

$$\begin{aligned}
c(\omega_s^-)+\tfrac{1}{20}\alpha_f 2^{-k}-\eta & \leqslant c(\omega_s^-)+\tfrac{1}{20}\alpha_f 2^{-k}-c(\omega_s^-)-\tfrac{9}{40}\alpha_f 2^{-k} \\
& = -\tfrac{7}{40}\alpha_f 2^{-k}, \\
c(\omega_s^-)-\tfrac{1}{20}\alpha_f 2^{-k}-\eta & \geqslant c(\omega_s^-)-\tfrac{1}{20}\alpha_f 2^{-k}-c(\omega_s^-)-\tfrac{31}{40}\alpha_f 2^{-k} \\
& = -\tfrac{33}{40}\alpha_f 2^{-k}.
\end{aligned}$$

It follows that $\widehat{\phi_s}(\lambda_n+\eta)=0$ for $\lambda_n \notin [-\frac{33}{40}\alpha_f 2^{-k}, -\frac{7}{40}\alpha_f 2^{-k}]$ and hence

$$|\langle\tau_y M_\eta f, \psi_{s,\tau_y M_\eta f}\rangle| \lesssim \sqrt{\alpha_f} \sum_{\substack{n\in\mathbb{Z} \\ \lambda_n \in [-\frac{33}{40}\alpha_f 2^{-k}, -\frac{7}{40}\alpha_f 2^{-k}]}} |\widehat{f}(\lambda_n)|.$$

Now, as f is a trigonometric polynomial, there exists $\lambda_{\max} \in \mathbb{R}^+$ such that $\widehat{f}(\lambda)=0$ for all λ with $|\lambda| > \lambda_{\max}$. It follows that for any k such that $2^{-k} > \max(\frac{40}{7}\alpha_f^{-1}\lambda_{\max}, 40\alpha_f^{-1}|\xi|)$,

$$\sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi+\eta)\langle\tau_y M_\eta f, \psi_{s,\tau_y M_\eta f}\rangle M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f} \equiv 0.$$

It is noted that the requirement that $\alpha_f 2^{-k} > 40|\xi|$ combined with the dyadic positioning of the tiles means that either $\xi < 0$ or there are no terms in the sum in s (this explains why the inequality on the inner product only considers negative values of λ_n). Either way, the above identity holds and it follows that for any fixed trigonometric polynomial, the full averaged operator consists of a sum of finitely many averaged operators at single scales. As such, it suffices to show existence of the averages at each individual scale.

Showing that the modulation average exists is straightforward. From the modulation symmetry described in Section 3.2, $M_{-\eta}A_{\rho,\xi+\eta,k}M_\eta f$ is periodic – and hence also almost periodic – in η for any fixed $k\in\mathbb{Z}$. It follows that the average in η exists.*

To show that the translation average exists, calculate as in Section 3.2 to observe that

$$\begin{aligned}
&\tau_{-y}A_{\rho,\xi,k}\tau_y f(x) \\
&= \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi)\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\frac{1}{|\Lambda_f|}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} e^{-2\pi i\lambda_n t}e^{-2\pi i\lambda_n y}\right. \\
&\qquad \left.\times\sum_{m\in\mathbb{Z}}\overline{\phi_{(I_s-y)\times(\omega_s-\lambda_n)}\big(t+\tfrac{m}{\alpha_f}\big)}\,dt\right)\frac{1}{|\Lambda_f|}\sum_{\substack{n'\in\mathbb{Z}\\ \lambda_{n'}\in\Lambda_f}} e^{2\pi i\lambda_{n'}x}e^{2\pi i\lambda_{n'}y} \\
&\qquad \times\sum_{m'\in\mathbb{Z}}\phi_{(I_s-y)\times(\omega_s-\lambda_{n'})}\big(x+\tfrac{m'}{\alpha_f}\big)
\end{aligned}$$

*It can be regarded as the calculation of an almost periodic Fourier coefficient.

$$
\begin{aligned}
&= \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \chi_{\omega_s^+}(\xi) \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} \Bigg(\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\frac{1}{|\Lambda_f|}e^{-2\pi i\lambda_n t} \\
&\qquad \times \sum_{m\in\mathbb{Z}} \overline{\phi_{(I_s-y)\times(\omega_s-\lambda_n)}\big(t+\tfrac{m}{\alpha_f}\big)\,dt} \Bigg) e^{-2\pi i\lambda_n y}\frac{1}{|\Lambda_f|} \sum_{\substack{n'\in\mathbb{Z}\\ \lambda_{n'}\in\Lambda_f}} \big(e^{2\pi i\lambda_{n'}x} \\
&\qquad \times \sum_{m'\in\mathbb{Z}} \phi_{(I_s-y)\times(\omega_s-\lambda_{n'})}\big(x+\tfrac{m'}{\alpha_f}\big)\big)e^{2\pi i\lambda_{n'}y} \\
&= \sum_{s\in\mathbb{D}_{\rho,\alpha_f,k}} \sum_{\substack{n,n'\in\mathbb{Z}\\ \lambda_n,\lambda_{n'}\in\Lambda_f}} \Bigg[\chi_{\omega_s^+}(\xi) \Bigg(\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\frac{1}{|\Lambda_f|}e^{-2\pi i\lambda_n t} \\
&\qquad \times \sum_{m\in\mathbb{Z}} \overline{\phi_{(I_s-y)\times(\omega_s-\lambda_n)}\big(t+\tfrac{m}{\alpha_f}\big)\,dt} \Bigg) \frac{1}{|\Lambda_f|}e^{2\pi i\lambda_{n'}x} \\
&\qquad \times \sum_{m'\in\mathbb{Z}} \phi_{(I_s-y)\times(\omega_s-\lambda_{n'})}\big(x+\tfrac{m'}{\alpha_f}\big) \Bigg] e^{2\pi i(\lambda_{n'}-\lambda_n)y}.
\end{aligned}
$$

Now, noting that $\phi_s(x) := |I_s|^{-\frac{1}{2}}\phi\left(\frac{x-c(I_s)}{|I_s|}\right)e^{2\pi i c(\omega_s^-)x}$, observe that for $z\in\mathbb{R}$,

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}} \phi_{(I_s-y)\times(\omega_s-\lambda_n)}\big(z+\tfrac{m}{\alpha_f}\big) \\
&= \sum_{m\in\mathbb{Z}} |I_s|^{-\frac{1}{2}}\phi\left(\frac{z+\frac{m}{\alpha_f}-c(I_s)+y}{|I_s|}\right)e^{2\pi i c(\omega_s^-)(z+\frac{m}{\alpha_f})} \\
&= \sum_{m\in\alpha_f^{-1}\mathbb{Z}+y} |I_s|^{-\frac{1}{2}}\phi\left(\frac{z+m-c(I_s)}{|I_s|}\right)e^{2\pi i c(\omega_s^-)(z+m-y)}.
\end{aligned}
$$

Since the exponential term here is periodic in y with period $(\min|c(\omega_s^-)|)^{-1} = 4\alpha_f^{-1}$ (given that $\max_{s\in\mathbb{D}_{\rho,\alpha_f}}|I_s| = \alpha_f^{-1}$ and all the possible choices for $c(\omega_s^-)$ are nonzero integer multiples of $\min|c(\omega_s^-)|$), it follows that $\sum_{m\in\mathbb{Z}} \phi_{(I_s-y)\times(\omega_s-\lambda_n)}(z+\frac{m}{\alpha_f})$ is periodic in y with period $4\alpha_f^{-1}$. The same is thus true of the entire expression in square brackets.

Clearly $e^{2\pi i(\lambda_{n'}-\lambda_n)y}$ is periodic in y, so each term in the triple outer sum from the above expression for $\tau_{-y}A_{\rho,\xi,k}\tau_y f(x)$ is almost periodic. Given that all three sums have finitely many terms, it follows that $\tau_{-y}A_{\rho,\xi,2^k}\tau_y$ is almost periodic in y and so the average in y exists.

In fact, more than this can be seen to be true. As the square-bracketed term is multiplied by only an exponential, $e^{2\pi i(\lambda_{n'}-\lambda_n)y}$, the averaging operation is equivalent to calculating Fourier coefficients of a periodic function. This ensures that the average will be zero except where there is a rational dependence between the period of the exponential term and that of the square-bracketed term. As such, without loss of generality, each term in the triple sum may be assumed to be periodic. Since the periods of the various terms will also be rationally dependent, the whole expression may be assumed to be periodic. This observation helps simplify calculations in what follows.

Whilst the existence of the modulation and translation averages has been established for all choices of $\rho \in \mathbb{R}$, the symmetry properties of the averaged operator are enhanced by setting ρ to 0 from this point onwards. The nature of these symmetry properties will be considered below. Since this choice removes a parameter from the averaged operator, the following definition is given:

$$\Pi_\xi f := \lim_{J,K,\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 M_{-\eta} \tau_{-y} A^{2^z}_{0,\xi+\eta} \tau_y M_\eta f \, dz \, dy \, d\eta.$$

3.3.2 Properties

In this section, various properties of the operator Π_ξ will be considered in turn. The purpose of this, aside from gaining a better understanding of this operator, is to parallel Lacey and Thiele's deduction from analogous properties in the $L^2(\mathbb{R})$ setting that the boundedness of the maximal averaged model operator is equivalent to the boundedness of the Carleson operator. Technicalities of the almost periodic form will require a small amount of simplification and omission here, the occurrence of which will be made clear with footnotes. The deductions of the present section will be made rigorous with a more direct series of calculations in Section 3.3.3.

To begin with, the modulation, translation and dilation symmetries that Π_ξ possesses will be considered. Throughout what follows, Theorem 1.3.6 will be used freely.

Modulation symmetry: Take any $\theta \in \mathbb{R}$. Then

$$\begin{aligned}
& M_{-\theta}\Pi_{\xi+\theta}M_\theta f \\
= & \lim_{J,K\to\infty}\frac{1}{4JK}\int_{-J}^{J}\int_{-K}^{K}\int_0^1 M_{-(\eta+\theta)}\tau_{-y}A^{2^z}_{0,\xi+\theta+\eta}\tau_y M_{(\eta+\theta)} f\,dz\,dy\,d\eta \\
= & \lim_{J,K\to\infty}\frac{1}{4JK}\int_{-J+\theta}^{J+\theta}\int_{-K}^{K}\int_0^1 M_{-\eta}\tau_{-y}A^{2^z}_{0,\xi+\eta}\tau_y M_\eta f\,dz\,dy\,d\eta \\
= & \lim_{J,K\to\infty}\frac{1}{4JK}\int_{-J}^{J}\int_{-K}^{K}\int_0^1 M_{-\eta}\tau_{-y}A^{2^z}_{0,\xi+\eta}\tau_y M_\eta f\,dz\,dy\,d\eta \\
= & \Pi_\xi f.
\end{aligned}$$

Translation symmetry: Take any $\theta \in \mathbb{R}$. Then

$$\begin{aligned}
& \tau_{-\theta}\Pi_\xi\tau_\theta f \\
= & \lim_{J,K\to\infty}\frac{1}{4JK}\int_{-J}^{J}\int_{-K}^{K}\int_0^1 \tau_{-\theta}M_{-\eta}\tau_{-y}A^{2^z}_{0,\xi+\eta}\tau_y M_\eta\tau_\theta f\,dz\,dy\,d\eta \\
= & \lim_{J,K\to\infty}\frac{1}{4JK}\int_{-J}^{J}\int_{-K}^{K}\int_0^1 e^{2\pi i\eta\theta}M_{-\eta}\tau_{-(y+\theta)}A^{2^z}_{0,\xi+\eta}e^{-2\pi i\eta\theta}\tau_{(y+\theta)}M_\eta f\,dz\,dy\,d\eta \\
= & \lim_{J,K\to\infty}\frac{1}{4JK}\int_{-J}^{J}\int_{-K+\theta}^{K+\theta}\int_0^1 M_{-\eta}\tau_{-y}A^{2^z}_{0,\xi+\eta}\tau_y M_\eta f\,dz\,dy\,d\eta \\
= & \Pi_\xi f.
\end{aligned}$$

Dilation symmetry: To allow a dilation symmetry without parameter dependence to hold, rather than considering Π_ξ here, the following operator will instead be the object of study:

$$S_\xi := M_{-\xi}\Pi_\xi M_\xi.$$

This operator inherits translation symmetry from Π_ξ.

Take any $\theta \in \mathbb{R}$. The perfect dilation symmetry of the model operator from Section 3.2 ($D_{\theta^{-1}} A_{0,\theta^{-1}\xi} D_\theta f = A_{0,\xi} f$) is independent of scale, so it is equally true that $D_{\theta^{-1}} A^{2^z}_{0,\theta^{-1}\xi} D_\theta f = A^{2^z}_{0,\xi} f$ for each $z \in [0,1]$. Consequently,

$$
\begin{aligned}
& D_{\theta^{-1}} S_\xi D_\theta \\
= \ & \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 D_{\theta^{-1}} M_{-(\xi+\eta)} \tau_{-y} A^{2^z}_{0,\xi+\eta} \tau_y M_{(\xi+\eta)} D_\theta f \, dz\, dy\, d\eta \\
= \ & \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 M_{-\theta(\xi+\eta)} D_{\theta^{-1}} \tau_{-y} A^{2^z}_{0,\xi+\eta} \tau_y D_\theta M_{\theta(\xi+\eta)} f \, dz\, dy\, d\eta \\
= \ & \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 M_{-\theta(\xi+\eta)} \tau_{-\theta^{-1}y} D_{\theta^{-1}} A^{2^z}_{0,\theta^{-1}(\theta(\xi+\eta))} D_\theta \tau_{\theta^{-1}y} M_{\theta(\xi+\eta)} f \, dz\, dy\, d\eta \\
= \ & M_{-\xi} M_{-(\theta-1)\xi} \Pi_{\xi+(\theta-1)\xi} M_{(\theta-1)\xi} M_\xi f \\
= \ & S_\xi f
\end{aligned}
$$

by the modulation symmetry on Π_ξ.

In addition to these three symmetry properties, it can further be shown that the averaged operator, acting on functions in $\mathscr{P}$, is B^2 bounded uniformly in ξ as a consequence of the boundedness of the model operator (Theorem 3.1.2). To do this, observe that for any $f \in \mathscr{P}$,

$$
\begin{aligned}
& \|\Pi_\xi f\|_{B^2} \\
= \ & \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \left| \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 M_{-\eta} \tau_{-y} A^{2^z}_{0,\xi+\eta} \tau_y M_\eta f(x) \, dz\, dy\, d\eta \right|^2 dx \right)^{\frac{1}{2}} \\
= \ & \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \left| \frac{1}{4J'K'} \int_{-J'}^{J'} \int_{-K'}^{K'} \int_0^1 M_{-\eta} \tau_{-y} A^{2^z}_{0,\xi+\eta} \tau_y M_\eta f(x) \, dz\, dy\, d\eta \right|^2 dx \right)^{\frac{1}{2}}
\end{aligned}
$$

for some fixed J' and K', by the aforementioned periodicity in the appropriate variables.* Using Minkowski's integral inequality, followed by the dominated convergence theorem (using the fact that f is a trigonometric polynomial), the previous expression is bounded by

$$
\frac{1}{4J'K'} \int_{-J'}^{J'} \int_{-K'}^{K'} \int_0^1 \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} |M_{-\eta} \tau_{-y} A^{2^z}_{0,\xi+\eta} \tau_y M_\eta f(x)|^2 \, dx \right)^{\frac{1}{2}} dz\, dy\, d\eta.
$$

*It is clear from the reasoning in Sections 3.2 and 3.3.1 that the periodicity does hold in this compound expression with the same period.

Finally, using the uniform boundedness of the rescaled model operator (Theorem 3.3.1, which is an immediate consequence of Theorem 3.1.2), that the modulation and translation operators are isometries on Besicovitch spaces as well as the fact that $|\Lambda_{\tau_y M_\eta f}| = |\Lambda_f|$, it can be concluded that

$$\|\Pi_\xi f\|_{B^2} \lesssim \frac{1}{4J'K'|\Lambda_f|^{\frac{3}{2}}} \int_{-J'}^{J'} \int_{-K'}^{K'} \int_0^1 \|f\|_{B^2}\, dz\, dy\, d\eta = \frac{1}{|\Lambda_f|^{\frac{3}{2}}} \|f\|_{B^2}$$

uniformly in $\xi \in \mathbb{R}$. It is noted that this uniform boundedness trivially extends to S_ξ.

Another property of S_ξ that can be deduced is that it vanishes when applied to trigonometric polynomials with frequency support in $[0,\infty)$ (that is to say functions $f \in \mathscr{P}$ such that $\sigma(f) \subseteq [0,\infty)$). To prove this, let f be such a function and noting that $\alpha_{\tau_y M_\eta f} = \alpha_f$ for any $y, \eta \in \mathbb{R}$, $z \in [0,1]$, recall that

$$A^{2^z}_{0,\xi+\eta} \tau_y M_{\xi+\eta} f = \sum_{s \in \mathbb{D}_{0,2^z \alpha_f}} \chi_{\omega_s^+}(\xi+\eta) \langle \tau_y M_{\xi+\eta} f, \psi_{s,\tau_y M_{\xi+\eta} f} \rangle \psi_{s,\tau_y M_{\xi+\eta} f}.$$

Observe also that

$$\langle \tau_y M_{\xi+\eta} f, \psi_{s,\tau_y M_{\xi+\eta} f} \rangle = \frac{\sqrt{\alpha_f}}{|\Lambda_{\tau_y M_{\xi+\eta} f}|} \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n \in \sigma(\tau_y M_{\xi+\eta} f) \cap \omega_s^-}} \widehat{f}(\lambda_n - \xi - \eta) e^{-2\pi i \lambda_n y} \overline{\widehat{\phi_s}(\lambda_n)}.$$

By the frequency support property of f, for each term in the sum in n that makes a non-zero contribution to the above expression, $\lambda_n - \xi - \eta \in [0,\infty)$, which implies that $\lambda_n \in [\xi+\eta, \infty)$. However, for any non-zero term in the sum in s, $\xi + \eta \in \omega_s^+$ which means that $\widehat{\phi_s}(\lambda_n) = 0$. This proves the claimed property.

It can also be shown that Π_ξ is positive semi-definite with respect to B^2 inner products:

$$\begin{aligned}
&\langle \Pi_\xi f, f \rangle \\
&= \Big\langle \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 \sum_{s \in \mathbb{D}_{0,2^z \alpha_f}} \chi_{\omega_s^+}(\xi+\eta) \langle \tau_y M_\eta f, \psi_{s,\tau_y M_\eta f} \rangle \\
&\qquad \times M_{-\eta} \tau_{-y} \psi_{s,\tau_y M_\eta f}\, dz\, dy\, d\eta, f \Big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J}\int_{-K}^{K}\int_{0}^{1} \sum_{s\in\mathbb{D}_{0,2^z\alpha_f}} \chi_{\omega_s^+}(\xi+\eta)\langle\tau_y M_\eta f, \psi_{s,\tau_y M_\eta f}\rangle \\
&\quad \times \langle M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f}, f\rangle \, dy\, d\eta \\
&= \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J}\int_{-K}^{K}\int_{0}^{1} \sum_{s\in\mathbb{D}_{0,2^z\alpha_f}} \chi_{\omega_s^+}(\xi+\eta)|\langle\tau_y M_\eta f, \psi_{s,\tau_y M_\eta f}\rangle|^2 \, dy\, d\eta \\
&\geqslant 0.
\end{aligned}
$$

As before, the interchange of limits can be justified by periodicity arguments. It also follows from the above that

$$
\left\langle \sum_{s\in\mathbb{D}_{0,2^z\alpha_f}} \chi_{\omega_s^+}(\xi+\eta)\langle\tau_y M_\eta f, \psi_{s,\tau_y M_\eta f}\rangle M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f}, f \right\rangle \geqslant 0
$$

for each $\eta, y \in \mathbb{R}$ and $z \in [0,1]$.

It can easily be shown that the model operator is a non-zero operator. Indeed, let $g(x) \coloneqq e^{2\pi i \frac{1}{4}x} + e^{2\pi i \frac{5}{4}x}$ and fix $\xi = \frac{3}{4}$. Here, $\alpha_g = 1$ and $\Lambda_g = \{\frac{1}{4}\}$. For the sake of clarity, it is emphasised that the second term in the definition of g is included purely to make the constant α_g well-defined. It will have essentially no other role in the following argument.

Consider that

$$
\begin{aligned}
A_{0,\frac{3}{4}} g(x) &= \sum_{s\in\mathbb{D}_{0,1}} \chi_{\omega_s^+}(\tfrac{3}{4})\langle g, \psi_{s,g}\rangle\psi_{s,g}(x) \\
&= \langle g, \psi_{[0,1]\times[0,1],g}\rangle\psi_{[0,1]\times[0,1],g}(x),
\end{aligned}
$$

as $[0,1]\times[0,1]$ is the only tile in $\mathbb{D}_{0,1}$ with $\frac{3}{4}\in\omega_s^+$ and $\frac{1}{4}\in\omega_s^-$ (there are clearly no tiles in $\mathbb{D}_{0,1}$ with $\frac{3}{4}\in\omega_s^+$ and $\frac{5}{4}\in\omega_s^-$). It follows that

$$
\begin{aligned}
A_{0,\frac{3}{4}} g(x) &= \overline{\widehat{\varphi}_{[0,1]\times[0,1]}(\tfrac{1}{4})} \sum_{m\in\mathbb{Z}} \widehat{\varphi}_{[0,1]\times[0,1]}(\tfrac{1}{4}+m)e^{2\pi i(\frac{1}{4}+m)x} \\
&= \left|\widehat{\varphi}_{[0,1]\times[0,1]}(\tfrac{1}{4})\right|^2 e^{\frac{1}{2}\pi i x},
\end{aligned}
$$

by the support properties of $\widehat{\phi}$. This shows that the unaveraged model operator, $A_{0,\frac{3}{4}}$, is non-zero (for any sensible choice of ϕ). It is also clear that for values of z sufficiently close to 0,

$$A^{2^z}_{0,\frac{3}{4}}g(x) = \left|\widehat{\phi}_{[0,2^{-z}]\times[0,2^z]}\left(\tfrac{1}{4}\right)\right|^2 e^{\frac{1}{2}\pi i x}.$$

The above argument adapts easily to $\xi \neq \frac{3}{4}$ and larger values of z and so it can be concluded that $A^{2^z}_{0,\xi}$ is a non-zero operator. That the averaged operator is also a non-zero operator can be shown in a similar way, but the formulation is somewhat more intricate. Since more general and direct calculations on Π_ξ will be carried out in Section 3.3.3, the details are omitted here.

To summarise the above, S_ξ has the following properties for all $\xi \in \mathbb{R}$:

- It is a non-zero operator.
- It is bounded uniformly on B^2.
- It commutes with translations and dilations.
- It vanishes when applied to functions with frequency support in $[0,\infty)$.

It is also the case that for each $f \in B^2$, the operator $\Pi_\xi f$ (and hence also $S_\xi f$) is of the form $\dfrac{1}{|\Lambda_f|^2} T_\xi f$ where T_ξ is a linear operator for each $\xi \in \mathbb{R}$.* This property seems to be difficult to show directly, but will be seen to be true as a consequence of the forthcoming calculations in Section 3.3.3.

The above properties can essentially be used to show that Π_ξ is, up to a constant multiple, equivalent to almost periodic Fourier summation and hence that $\sup_{\xi\in\mathbb{R}} |\Pi_\xi \cdot|$ is essentially

*Technically, this is only true for Π_ξ if $\xi - \lambda_n > \frac{2\alpha_f}{5}$ for all $\lambda_n \in \sigma(f) \cap (-\infty, \xi]$ and only true for S_ξ if $\lambda_n < -\frac{2\alpha_f}{5}$ for all $\lambda_n \in \sigma(f) \cap (-\infty, 0]$. These restrictions will be disregarded for the remainder of this section to allow for a clearer exposition of the fundamental ideas. Further details will be provided in Section 3.3.3.

the same operator as the Carleson operator. Firstly, by B^2 boundedness and translation commutation, Theorem 1.3.14 provides that for any $f \in \mathscr{P}$,

$$S_\xi f = \frac{1}{|\Lambda_f|^2} \sum_{n \in \mathbb{Z}} m_\xi(\lambda_n) \widehat{f}(\lambda_n) e^{2\pi i \lambda_n \cdot}$$

for some $m_\xi \in L^\infty(\mathbb{R})$.*

By applying S_ξ to appropriate functions, the vanishing property gives that $m_\xi(\lambda) = 0$ for any $\lambda \geqslant 0$. Dilation commutation gives that for any $z \in \mathbb{R}$,

$$S_\xi(f(z\cdot))(z^{-1}x) = S_\xi f(x).$$

However, calculating directly,

$$S_\xi(f(z\cdot))(z^{-1}x) = \frac{1}{|\Lambda_f|^2} \sum_{n \in \mathbb{Z}} m_\xi(\lambda_n) \widehat{f}(z^{-1}\lambda_n) e^{2\pi i (z^{-1}\lambda_n)x} = \frac{1}{|\Lambda_f|^2} \sum_{n \in \mathbb{Z}} m_\xi(z\lambda_n) \widehat{f}(\lambda_n) e^{2\pi i \lambda_n x}.$$

As such, it follows that there exists a constant $c_\xi \in \mathbb{R}$ such that $m_\xi(\lambda) = c_\xi$ for all $\lambda < 0$. Further, this constant must be non-zero since S_ξ is a non-zero operator. This shows that

$$S_\xi f = \frac{c_\xi}{|\Lambda_f|^2} \sum_{\lambda_n < 0} \widehat{f}(\lambda_n) e^{2\pi i \lambda_n \cdot}.$$

Recalling that $S_\xi \coloneqq M_{-\xi} \Pi_\xi M_\xi$ and applying S_ξ to $M_{-\xi} f$ it follows that

$$\Pi_\xi f = \frac{c_\xi}{|\Lambda_f|^2} \sum_{\lambda_n < \xi} \widehat{f}(\lambda_n) e^{2\pi i \lambda_n \cdot}.$$

*There is another technicality to be avoided here, namely that the model operator, and hence also S_ξ, is not defined on single exponentials, $g = e^{i\lambda\cdot}$ (since α_g is then undefined). The proof of Theorem 1.3.14 relies on building up the operator from its constituent parts acting on single exponentials, so it is not immediately applicable here. This issue can be circumvented by making a canonical choice, $\alpha_g = |\lambda|$, for example (for $\lambda \neq 0$; $S_\xi f$ is not defined on trigonometric polynomials with a constant term owing to the requirement that $\lambda_n < -\frac{2\alpha_f}{5}$ for all $\lambda_n \in \sigma(f) \cap (-\infty, 0]$). A constant choice for α_g is not sufficient as it does not respect the dilation symmetry of S_ξ.

Using this, for any fixed $\theta \in \mathbb{R}$,

$$\begin{aligned} M_{-\theta}\Pi_{\xi+\theta}M_{\theta}f &= e^{-2\pi i\theta x}\frac{c_{\xi+\theta}}{|\Lambda_f|^2}\sum_{\lambda_n<\xi+\theta}\widehat{f}(\lambda_n-\theta)e^{2\pi i\lambda_n\cdot} \\ &= \frac{c_{\xi+\theta}}{|\Lambda_f|^2}\sum_{\lambda_n<\xi}\widehat{f}(\lambda_n)e^{2\pi i\lambda_n\cdot}. \end{aligned}$$

The modulation symmetry property of Π_ξ implies that $M_{-\theta}\Pi_{\xi+\theta}M_\theta f = \Pi_\xi f$, so it follows that $c_{\xi+\theta} = c_\xi$ for any $\theta \in \mathbb{R}$, that is to say that c_ξ does not depend on ξ. It may thus be concluded that

$$\sup_{\xi\in\mathbb{R}}|\Pi_\xi f| \approx \frac{1}{|\Lambda_f|^2}\mathscr{C}f$$

and so $\sup_{\xi\in\mathbb{R}}|\Pi_\xi\,\cdot\,|$ is a good model for the Carleson operator on $\mathscr{P}$ in the sense that its (weak) B^2 boundedness is equivalent to that of $\frac{1}{|\Lambda_f|^2}\mathscr{C}$.*

3.3.3 The Question of Linearity and Direct Calculation

In this section, the fact that

$$\Pi_\xi f \approx \frac{1}{|\Lambda_f|^2}\sum_{\lambda_n<\xi}\widehat{f}(\lambda_n)e^{2\pi i\lambda_n\cdot}$$

for all ξ in

$$\Xi_f := \{\xi\in\mathbb{R} : \xi-\lambda_n > \tfrac{2\alpha_f}{5}\ \forall\,\lambda_n\in\sigma(f)\cap(-\infty,\xi]\}$$

will be shown by direct calculation. It was claimed above that for each $f \in B^2$, $\Pi_\xi f$ (and hence also $S_\xi f$) is of the form $\frac{1}{|\Lambda_f|^2}T_\xi f$ where T_ξ is a linear operator for each $\xi \in \mathbb{R}$. This

*Given the restriction that $\xi-\lambda_n > \frac{2\alpha_f}{5}$ for all $\lambda_n \in \sigma(f)\cap(-\infty,\xi]$, strictly speaking, it is necessary to take the supremum over this restricted choice of ξ here, although it is certainly true that $\frac{1}{|\Lambda_f|^2}\mathscr{C}f \lesssim \sup_{\xi\in\mathbb{R}}|\Pi_\xi f|$.

follows from this calculation, although it will, of course, no longer be necessary to apply this fact to the reasoning above.

To begin with fix any $f \in \mathscr{P}$ and recall that for each $\xi \in \mathbb{R}$, $\Pi_\xi f$ is equal to

$$\lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 \sum_{s\in\mathbb{D}_{0,2^z\alpha_f}} \chi_{\omega_s^+}(\xi+\eta) \langle\!\langle \tau_y M_\eta f, \psi_{s,\tau_y M_\eta f} \rangle\!\rangle M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f}\, dz\, dy\, d\eta$$

which, splitting the sum in s into its constituent scales, is equal to

$$\sum_{k=-\infty}^{0} \lim_{J,K\to\infty} \frac{1}{4JK} \int_{-J}^{J} \int_{-K}^{K} \int_0^1 \sum_{\substack{s\in\mathbb{D}_{0,2^z\alpha_f}\\ |I_s|=\frac{2^{k-z}}{\alpha_f}}} \chi_{\omega_s^+}(\xi+\eta) \langle\!\langle f, M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f} \rangle\!\rangle \times M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f}\, dz\, dy\, d\eta.$$

Now,

$$\begin{aligned}
&\psi_{s,\tau_y M_\eta f}\\
=\ & \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_{\tau_y M_\eta f}}} \sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n+\alpha_f m) e^{2\pi i(\lambda_n+\alpha_f m)\cdot}\\
=\ & \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} \sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n+\eta+\alpha_f m) e^{2\pi i(\lambda_n+\eta+\alpha_f m)\cdot}.
\end{aligned}$$

Consequently,

$$M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f}(x) = \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}} \sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n+\eta+\alpha_f m) e^{2\pi i(\lambda_n+\eta+\alpha_f m)y} e^{2\pi i(\lambda_n+\alpha_f m)\cdot}.$$

It follows that

$$\langle\!\langle f, M_{-\eta}\tau_{-y}\psi_{s,\tau_y M_\eta f} \rangle\!\rangle = \frac{\sqrt{\alpha_f}}{|\Lambda_f|} \sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma(f)}} \widehat{f}(\lambda_n) e^{-2\pi i(\lambda_n+\eta)y} \overline{\widehat{\phi_s}(\lambda_n+\eta)},$$

so

$$\begin{aligned}
&\Pi_\xi f \\
=\;& \frac{\alpha_f}{|\Lambda_f|^2}\sum_{k=-\infty}^{0}\lim_{J,K\to\infty}\frac{1}{4JK}\int_{-J}^{J}\int_{-K}^{K}\int_0^1 \sum_{\substack{s\in\mathbb{D}_{0,2^z\alpha_f}\\ |I_s|=\frac{2^{k-z}}{\alpha_f}}}\chi_{\omega_s^+}(\xi+\eta)\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma(f)}}\widehat{f}(\lambda_n)e^{-2\pi i(\lambda_n+\eta)y} \\
&\times\overline{\widehat{\phi_s}(\lambda_n+\eta)}\sum_{\substack{n'\in\mathbb{Z}\\ \lambda_{n'}\in\Lambda_f}}\sum_{m\in\mathbb{Z}}\widehat{\phi_s}(\lambda_{n'}+\eta+\alpha_f m)e^{2\pi i(\lambda_{n'}+\eta+\alpha_f m)y}e^{2\pi i(\lambda_{n'}+\alpha_f m)\cdot}\,dz\,dy\,d\eta \\
=\;& \frac{\alpha_f}{|\Lambda_f|^2}\sum_{k=-\infty}^{0}\lim_{J\to\infty}\frac{1}{2J}\int_{-J}^{J}\int_0^1 \sum_{\substack{s\in\mathbb{D}_{0,2^z\alpha_f}\\ |I_s|=\frac{2^{k-z}}{\alpha_f}}}\chi_{\omega_s^+}(\xi+\eta)\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma(f)}}\sum_{\substack{n'\in\mathbb{Z}\\ \lambda_{n'}\in\Lambda_f}}\sum_{m\in\mathbb{Z}}\widehat{f}(\lambda_n)\overline{\widehat{\phi_s}(\lambda_n+\eta)} \\
&\times\widehat{\phi_s}(\lambda_{n'}+\eta+\alpha_f m)\left(\lim_{K\to\infty}\frac{1}{2K}\int_{-K}^{K}e^{2\pi i(\lambda_{n'}+\alpha_f m-\lambda_n)y}\,dy\right)e^{2\pi i(\lambda_{n'}+\alpha_f m)\cdot}\,dz\,d\eta.
\end{aligned}$$

Using orthonormality of exponentials with respect to Besicovitch inner products, this can be seen to be equal to

$$\frac{\alpha_f}{|\Lambda_f|^2}\sum_{k=-\infty}^{0}\lim_{J\to\infty}\frac{1}{2J}\int_{-J}^{J}\int_0^1 \sum_{\substack{s\in\mathbb{D}_{0,2^z\alpha_f}\\ |I_s|=\frac{2^{k-z}}{\alpha_f}}}\chi_{\omega_s^+}(\xi+\eta)\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma(f)}}\widehat{f}(\lambda_n)|\widehat{\phi_s}(\lambda_n+\eta)|^2e^{2\pi i\lambda_n\cdot}\,dz\,d\eta.$$

In other words,

$$\Pi_\xi f=\frac{1}{|\Lambda_f|^2}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\sigma(f)}}\widehat{f}(\lambda_n)C_{f,\lambda_n,\xi}e^{2\pi i\lambda_n\cdot},$$

where

$$C_{f,\lambda_n,\xi}=\alpha_f\sum_{k=-\infty}^{0}\lim_{J\to\infty}\frac{1}{2J}\int_{-J}^{J}\int_0^1\sum_{\substack{s\in\mathbb{D}_{0,\alpha_f}\\ |I_s|=\frac{2^k}{\alpha_f}}}\chi_{2^z\omega_s^+}(\xi+\eta)|\widehat{\phi}_{2^{-z}I_s\times 2^z\omega_s}(\lambda_n+\eta)|^2\,dz\,d\eta.$$

By periodicity in η at each scale, $C_{f,\lambda_n,\xi}$ is equal to

$$\sum_{k=-\infty}^{0}\int_0^1 2^{k-z}\int_{\mathscr{I}_{k,z,f}}\sum_{\substack{s\in\mathbb{D}_{0,\alpha_f}\\|I_s|=\frac{2^k}{\alpha_f}}}\chi_{\omega_s^+}(2^{-z}(\xi+\eta))|\widehat{\phi_s}(2^{-z}(\lambda_n+\eta))|^2\,dz\,d\eta$$

$$=\sum_{k=-\infty}^{0}\sum_{\substack{s\in\mathbb{D}_{0,\alpha_f}\\|I_s|=\frac{2^k}{\alpha_f}}}\int_0^1 2^{k-z}\int_{\mathscr{I}_{k,z,f}}\chi_{\omega_s^+}(2^{-z}(\xi+\eta))|\widehat{\phi_s}(2^{-z}(\lambda_n+\eta))|^2\,dz\,d\eta,$$

where $\mathscr{I}_{k,z,f}$ is any interval of length $\alpha_f 2^{z-k}$. This in turn is equal to

$$\sum_{k=-\infty}^{0}\sum_{\substack{s\in\mathbb{D}_{0,\alpha_f}\\|I_s|=\frac{2^k}{\alpha_f}}}\int_0^1 2^{k}\int_{\widetilde{\mathscr{I}}_{k,z,f}}\chi_{\omega_s^+}(2^{-z}\xi+\eta)|\widehat{\phi_s}(2^{-z}\lambda_n+\eta)|^2\,dz\,d\eta$$

$$=\sum_{k=-\infty}^{0}\sum_{\substack{s\in\mathbb{D}_{0,\alpha_f}\\|I_s|=\frac{2^k}{\alpha_f}}}2^k\int_0^1\int_{\widetilde{\mathscr{I}}_{k,z,f}\cap(\omega_s^+-2^{-z}\xi)\cap(\frac{1}{5}\star\omega_s^--2^{-z}\lambda_n)}|\widehat{\phi}_{I\times(\omega_s-2^{-z}\lambda_n)}(\eta)|^2\,d\eta\,dz,$$

where $\widetilde{\mathscr{I}}_{k,z,f}$ is any interval of length $\alpha_f 2^{-k}$ (which may be chosen differently for each $z\in[0,1]$) and I is *any* interval (the choice of which is unimportant since $|\widehat{\phi}_{I\times\omega}|$ does not depend on I).

For each z, choose $\widetilde{\mathscr{I}}_{k,z,f}$ such that there exists a *unique* ω_{s_k} corresponding to tiles from the sum in s such that $(\omega_{s_k}^+-2^{-z}\xi)\cap\widetilde{\mathscr{I}}_{k,z,f}\neq\emptyset$. By the positioning of dyadic tiles and the fact that $|\widetilde{\mathscr{I}}_{k,z,f}|=\alpha_f 2^{-k}$, it is necessarily the case that $(\omega_{s_k}^+-2^{-z}\xi)\subseteq\widetilde{\mathscr{I}}_{k,z,f}$.

Since there are 2^{-k} choices of s with $\omega_s=\omega_{s_k}$, it follows that

$$C_{f,\lambda_n,\xi}=\int_0^1\sum_{k=-\infty}^{0}\int_{(\omega_{s_k}^+-2^{-z}\xi)\cap(\frac{1}{5}\star\omega_{s_k}^--2^{-z}\lambda_n)}|\widehat{\phi}_{I\times(\omega_{s_k}-2^{-z}\lambda_n)}(\eta)|^2\,d\eta\,dz.$$

The inner integral here is exactly $\|\phi\|^2_{L^2(\mathbb{R})}$ whenever $(\frac{1}{5}\star\omega_{s_k}^--2^{-z}\lambda_n)\subseteq(\omega_{s_k}^+-2^{-z}\xi)$. This happens precisely when $2^{-z}(\xi-\lambda_n)\in[\frac{3|\omega_{s_k}|}{10},\frac{7|\omega_{s_k}|}{10}]$. On the other hand, the inner integral is zero whenever $(\frac{1}{5}\star\omega_{s_k}^--2^{-z}\lambda_n)\cap(\omega_{s_k}^+-2^{-z}\xi)=\emptyset$, and up to the end-point, this happens precisely when $2^{-z}(\xi - \lambda_n) \notin (\frac{|\omega_{s_k}|}{5}, \frac{4|\omega_{s_k}|}{5})$. It follows in particular that the sum in k has at most two non-zero terms for any given z.

Now, fix any $\xi, \lambda_n \in \mathbb{R}$ with $\xi - \lambda_n > \frac{2\alpha_f}{5}$. Then for any $z \in [0,1]$, there exists a unique choice of $k^* \in (-\infty, 0] \cap \mathbb{Z}$ such that $2^{-z}(\xi - \lambda_n) \in (\frac{|\omega_{s_{k^*}}|}{5}, \frac{2|\omega_{s_{k^*}}|}{5}]$. It can be seen that the quantity

$$\sum_{k=-\infty}^{0} \int_{(\omega_{s_k}^{+} - 2^{-z}\xi) \cap (\frac{1}{5}\star\omega_{s_k}^{-} - 2^{-z}\lambda_n)} |\widehat{\phi}_{I \times (\omega_{s_k} - 2^{-z}\lambda_n)}(\eta)|^2 \, d\eta$$

depends only on the value of $\dfrac{2^{-z}(\xi - \lambda_n)}{|\omega_{s_{k^*}}|}$. Since the integral in z averages over all possible choices of this quantity, it can be concluded that $C_{f,\lambda_n,\xi}$ does not depend on f, λ_n or ξ and thus it is indeed the case that

$$\Pi_\xi f \approx \frac{1}{|\Lambda_f|^2} \sum_{\substack{n \in \mathbb{Z} \\ \lambda_n < \xi}} \widehat{f}(\lambda_n) e^{2\pi i \lambda_n \cdot}$$

for $\xi \in \Xi_f$ and thus

$$\frac{1}{|\Lambda_f|^2} \mathscr{C} f \approx \sup_{\xi \in \Xi_f} |\Pi_\xi f| \leqslant \sup_{\xi \in \mathbb{R}} |\Pi_\xi f|.$$

3.4 Reduction to the Main Estimate

It has been established that to deduce weak B^2 boundedness of the Carleson operator acting on trigonometric polynomials, it will suffice to show that

$$\| \sup_{\xi \in \mathbb{R}} |\Pi_\xi f| \|_{B^{2,\infty}} \lesssim \frac{1}{|\Lambda_f|^2} \|f\|_{B^2}$$

for all $f \in \mathscr{P}$. In this section, a number of reductions will be made that will show that it suffices to obtain a simpler inequality. This inequality will be referred to as the "main estimate".

The desired weak B^2 bound for the Carleson operator is that given in Theorem 2.1.1, namely that for any $f \in B^2$,

$$\lim_{T\to\infty} \frac{1}{2T} |\{x \in [-T,T] : |\mathscr{C} f(x)| > t\}| \lesssim \left(\frac{\|f\|_{B^2}}{t}\right)^2$$

for all $t > 0$, where $\mathscr{C}$ has been extended to B^2. The reduction of this inequality to the "main estimate" will be considered for $f \in \mathscr{P}$ only, since the time-frequency model construction is, in general, only meaningful for such functions. This will later be seen to be sufficient given a density argument for the Carleson operator.

To begin the aforementioned reductions, note that for any $f \in \mathscr{P}$, using the periodicity in the appropriate variables discussed in Section 3.3, it can be seen that

$$\|\sup_{\xi\in\mathbb{R}} |\Pi_\xi f|\|_{B^{2,\infty}} = \left\|\sup_{\xi\in\mathbb{R}} \left|\frac{1}{4J'K'} \int_{-J'}^{J'} \int_{-K'}^{K'} \int_0^1 M_{-\eta} \tau_{-y} A^{2^z}_{0,\xi+\eta} \tau_y M_\eta f \, dz \, dy \, d\eta\right|\right\|_{B^{2,\infty}}$$

for some appropriate J', $K' > 0$. From Proposition 1.3.13, the weak B^2 quasinorm may be replaced with a norm, so Minkowski's integral inequality may be applied to conclude that it is sufficient to show that

$$\sup_{t>0} t \left(\lim_{T\to\infty} \frac{1}{2T} |\{x \in [-T,T] : \sup_{\xi\in\mathbb{R}} |A^{2^z}_{0,\xi} f(x)| > t\}|\right)^{\frac{1}{2}} \lesssim \frac{1}{|\Lambda_f|^2} \|f\|_{B^2}$$

uniformly in $z \in [0,1]$. In fact, the subsequent analysis will be applied to the original maximal model operator, $\sup_{\xi\in\mathbb{R}} |A_{0,\xi} \cdot|$. It will be apparent this analysis is equally valid for the rescaled maximal operator above and that the desired uniform control is attainable.

Following Fefferman's approach (also used by Lacey and Thiele), as described in Section 1.1, the maximal operator may be linearised by choosing a measurable function $N_f : \mathbb{R} \to \mathbb{R}$ for each $f \in \mathscr{P}$ that selects the values of ξ where the supremum is essentially attained in the sense that for all $x \in \mathbb{R}$,

$$\sup_{\xi\in\mathbb{R}} |A_{0,\xi} f(x)| \leqslant 2|A_{0,N_f(x)} f(x)|.$$

Removing the dependence on f in this linearising function, it suffices to show that for any measurable function $N : \mathbb{R} \to \mathbb{R}$,

$$\sup_{t>0} t \left(\lim_{T\to\infty} \frac{1}{2T} |\{x \in [-T,T] : |A_{0,N} f(x)| > t\}| \right)^{\frac{1}{2}} \lesssim \frac{1}{|\Lambda_f|^2} \|f\|_{B^2}$$

across all $f \in \mathscr{P}$, with the estimate independent of the choice of N.

Now, recall that the function μ, acting on appropriate measurable sets $E \subseteq \mathbb{R}$, is defined as

$$\mu(E) \coloneqq \lim_{T\to\infty} \frac{1}{2T} |E \cap [-T,T]|$$

where this limit exists. Fix the function $N : \mathbb{R} \to \mathbb{R}$ and some arbitrary finite collection of tiles, $\mathbb{P}_f \subseteq \mathbb{D}_{0,\alpha_f}$.* Suppose that

$$\boxed{\sum_{s\in\mathbb{P}_f} |\langle\!\langle (\chi_{\omega_{s+}} \circ N)\psi_{s,f}, \chi_E \rangle\!\rangle \langle\!\langle f, \psi_{s,f} \rangle\!\rangle| \lesssim \frac{1}{|\Lambda_f|^2} \mu(E)^{\frac{1}{2}} \|f\|_{B^2}}$$

holds for all $f \in \mathscr{P}$ and for any measurable $E \subseteq \mathbb{R}$ such that $\mu(E)$ exists and is non-zero, with the estimate independent of $\mathbb{P}_f$, E and N. It will be shown that this supposition implies the weak B^2 boundedness of $A_{0,N}$.

To aid notation, the following operator is defined for $\mathbb{P}_f$ and N as described above:

$$A_{0,N}^{\mathbb{P}_f} f \coloneqq \sum_{s\in\mathbb{P}_f} \chi_{\omega_{s+}}(N(x)) \langle\!\langle f, \psi_{s,f} \rangle\!\rangle \psi_{s,f}.$$

For any fixed $t \in \mathbb{R}^+$, define $E_t \coloneqq \{x \in \mathbb{R} : |A_{0,N}^{\mathbb{P}_f} f(x)| > t\}$ and suppose for the sake of simplicity that $\mu(E_t) > 0$. Then, defining $(A_{0,N}^{\mathbb{P}_f} f)^{\pm} \coloneqq \{x \in \mathbb{R} : \pm A_{0,N}^{\mathbb{P}_f} f(x) > 0\}$, it can be

*It should be clarified that the subscript on $\mathbb{P}_f$ only refers to a dependence on α_f, which determines the location and scale of the tiles.

seen by Chebyshev's inequality and elementary manipulations that

$$\begin{aligned}
t\mu(E_t) &\leqslant \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|A_{0,N}^{\mathbb{P}_f}f(x)|\chi_{E_t}(x)\,dx \\
&\leqslant \Big|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A_{0,N}^{\mathbb{P}_f}f(x)\chi_{E_t\cap(A_{0,N}^{\mathbb{P}_f}f)^+}(x)\,dx\Big| \\
&\qquad +\Big|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A_{0,N}^{\mathbb{P}_f}f(x)\chi_{E_t\cap(A_{0,N}^{\mathbb{P}_f}f)^-}(x)\,dx\Big| \\
&\leqslant \sum_{s\in\mathbb{P}_f}|\langle\!\langle(\chi_{\omega_{s+}}\circ N)\psi_{s,f},\chi_{E_t\cap(A_{0,N}^{\mathbb{P}_f}f)^+}\rangle\!\rangle\langle\!\langle f,\psi_{s,f}\rangle\!\rangle| \\
&\qquad +\sum_{s\in\mathbb{P}_f}|\langle\!\langle(\chi_{\omega_{s+}}\circ N)\psi_{s,f},\chi_{E_t\cap(A_{0,N}^{\mathbb{P}_f}f)^-}\rangle\!\rangle\langle\!\langle f,\psi_{s,f}\rangle\!\rangle| \\
&\lesssim_{\alpha_f} \frac{1}{|\Lambda_f|^2}\big(\mu(E_t\cap(A_{0,N}^{\mathbb{P}_f}f)^+)^{\frac{1}{2}}+\mu(E_t\cap(A_{0,N}^{\mathbb{P}_f}f)^-)^{\frac{1}{2}}\big)\|f\|_{B^2} \\
&\qquad \text{by the supposition} \\
&\lesssim \frac{1}{|\Lambda_f|^2}\mu(E_t)^{\frac{1}{2}}\|f\|_{B^2}.
\end{aligned}$$

By independence from the choice of $\mathbb{P}_f$, it follows that

$$\sup_{t>0} t\left(\lim_{T\to\infty}\frac{1}{2T}|\{x\in[-T,T]:|A_{0,N}f(x)|>t\}|\right)^{\frac{1}{2}} \lesssim \frac{1}{|\Lambda_f|^2}\|f\|_{B^2}.$$

This is precisely the estimate needed to show weak B^2 boundedness of the Carleson operator acting on trigonometric polynomials.

Taking this estimate as established, to extend the Carleson operator from the class of trigonometric polynomials to the whole of B^2, first define the space $B^{2,\infty}$ to be the completion of $\mathscr{P}$ under the $B^{2,\infty}$ quasi-norm. Taking functions that are equivalent under this norm to be equal and recalling that the $B^{2,\infty}$ quasi-norm is equivalent to a norm (Proposition 1.3.13), this space can be considered to be a Banach space.

For any $f\in B^2$, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of trigonometric polynomials approximating f in the sense that $\lim_{n\to\infty}\|f-f_n\|_{B^2}=0$. Using the triangle inequality, it can be seen that for any n, $m\in\mathbb{N}$,

$$\|\mathscr{C}f_n-\mathscr{C}f_m\|_{B^{2,\infty}} \leqslant \|\mathscr{C}(f_n-f_m)\|_{B^{2,\infty}}$$

$$\lesssim \quad \|f_n - f_m\|_{B^2},$$

by the assumption of weak B^2 boundedness of the Carleson operator. It thus follows that $(\mathscr{C} f_n)_{n\in\mathbb{Z}}$ is a Cauchy sequence in $B^{2,\infty}$ and so by completeness, $\mathscr{C}$ can be continuously extended to the whole of B^2. It follows analogously that the weak B^2 boundedness of the Carleson operator acting on trigonometric polynomials extends to the whole of B^2.

3.5 Establishing the Main Estimate

From the deductions in the previous section, to prove Theorem 2.1.1, it will suffice to show the following:

Conjecture 3.5.1 *Let $E \subseteq \mathbb{R}$ be a measurable set such that $\mu(E)$ exists and is non-zero, let $N : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function, let f be a trigonometric polynomial and let $\mathbb{P}_f$ be an arbitrary finite collection of tiles from $\mathbb{D}_{0,\alpha_f}$. Then*

$$\boxed{\sum_{s\in\mathbb{P}_f} |\langle\!\langle (\chi_{\omega_s^+} \circ N)\psi_{s,f}, \chi_E\rangle\!\rangle \langle\!\langle f, \psi_{s,f}\rangle\!\rangle| \lesssim \frac{1}{|\Lambda_f|^2}\mu(E)^{\frac{1}{2}}\|f\|_{B^2}.}$$

Here, the following weaker result will be established:

Theorem 3.5.2 *Let $E \subseteq \mathbb{R}$ be a measurable set such that $\mu(E)$ exists and is non-zero, let $N : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function, let f be a trigonometric polynomial and let $\mathbb{P}_f$ be an arbitrary finite collection of tiles from $\mathbb{D}_{0,\alpha_f}$. Then*

$$\boxed{\sum_{s\in\mathbb{P}_f} |\langle\!\langle (\chi_{\omega_s^+} \circ N)\psi_{s,f}, \chi_E\rangle\!\rangle \langle\!\langle f, \psi_{s,f}\rangle\!\rangle| \lesssim \frac{1}{|\Lambda_f|^{\frac{1}{2}}}\mu(E)^{\frac{1}{2}}\|f\|_{B^2}.}$$

Throughout what follows, the set $E \subseteq \mathbb{R}$, functions $f \in B^2$, $N : \mathbb{R} \to \mathbb{R}$ and collection $\mathbb{P}_f \subseteq \mathbb{D}_{0,\alpha_f}$ will be taken to be fixed.

As the form of the main estimate (with either power of factor $\frac{1}{|\Lambda_f|}$) is analogous to the one that Lacey and Thiele establish to prove Carleson's Theorem on $L^2(\mathbb{R})$ (given at the end of Section 1.2.1), the approach used to prove Theorem 3.5.2 and that suggested for proving Conjecture 3.5.1 will follow a similar scheme. This requires concepts of "mass" and "energy" adapted to this context. The former can be defined as follows for a tile $s \in \mathbb{D}_{0,\alpha_f}$ and an arbitrary collection $\mathbb{S} \subseteq \mathbb{D}_{0,\alpha_f}$:

$$\mathcal{M}(E;\{s\}) := \mu(E)^{-1} \sup_{\substack{u \in \mathbb{D}_{0,\alpha_f} \\ s<u}} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E\cap N^{-1}(\omega_u)} \sum_{m\in\mathbb{Z}} \frac{|I_u|^{-1}}{\left(1+\frac{|x+\frac{m}{\alpha_f}-c(I_u)|}{|I_u|}\right)^{10}} \, dx,$$

$$\mathcal{M}(E;\mathbb{S}) := \sup_{s\in\mathbb{S}} \mathcal{M}(E;\{s\}).$$

As with the $L^2(\mathbb{R})$ concept of mass, the first expression is a generalisation of a natural expression of density of E with respect to a tile s. Here, an integrand that reflects the decay of $\psi_{s,f}$ is chosen, based on the estimate from Proposition 2.3.3.

The concept of energy in this context is a straightforward generalisation of the $L^2(\mathbb{R})$ expression. For a collection of tiles, $\mathbb{S} \subseteq \mathbb{D}_{0,\alpha_f}$, define

$$\mathcal{E}(f;\mathbb{S}) := \frac{1}{\|f\|_{B^2}} \sup_{\substack{\mathbb{T} \text{ a +tree} \\ \mathbb{T}\subseteq\mathbb{S}}} \left(\frac{1}{|I_{\mathrm{top}(\mathbb{T})}|} \sum_{s\in\mathbb{T}} |\langle\!\langle f, \psi_{s,f} \rangle\!\rangle|^2 \right)^{\frac{1}{2}}.$$

Throughout the remainder of this part of this thesis, for a tree $\mathbb{T}_j$ from an indexed collection, it will be written that $\mathrm{top}(\mathbb{T}_j) = I_j \times \omega_j$. Whilst this is an abuse of the notation scheme already introduced, it will serve to simplify the presentation in what follows.

The proof of Theorem 3.5.2 will proceed by using the following three lemmata:

The Mass Lemma *Let f be a trigonometric polynomial and let $N : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function. Choose $E \subseteq \mathbb{R}$ to be a measurable set such that $\mu(E)$ exists and is non-zero and let $\mathbb{P}_f$ be an arbitrary finite collection of tiles from $\mathbb{D}_{0,\alpha_f}$. Then $\mathbb{P}_f$ can be written as $\mathbb{P}_f^{\mathrm{light}} \sqcup \mathbb{P}_f^{\mathrm{heavy}}$ where*

$$\mathcal{M}(E;\mathbb{P}_f^{\mathrm{light}}) \leqslant \frac{1}{4}\mathcal{M}(E;\mathbb{P}_f)$$

and $\mathbb{P}_f^{\text{heavy}}$ *is a union of trees* $\mathbb{T}_j$ *with*

$$\sum_j |I_j| \lesssim \mathcal{M}(E;\mathbb{P}_f)^{-1}.$$

The Energy Lemma *Let* f *be a trigonometric polynomial and let* $\mathbb{P}_f$ *be an arbitrary finite collection of tiles from* $\mathbb{D}_{0,\alpha_f}$. *Then* $\mathbb{P}_f$ *can be written as* $\mathbb{P}_f^{\text{low}} \sqcup \mathbb{P}_f^{\text{high}}$ *where*

$$\mathcal{E}(f;\mathbb{P}_f^{\text{low}}) \leqslant \frac{1}{2}\mathcal{E}(f;\mathbb{P}_f)$$

and $\mathbb{P}_f^{\text{high}}$ *is a union of trees* $\mathbb{T}_j$ *such that*

$$\sum_j |I_j| \lesssim \frac{1}{|\Lambda_f|}\mathcal{E}(f;\mathbb{P}_f)^{-2}.$$

The Tree Lemma *Let* f *be a trigonometric polynomial, let* $E \subseteq \mathbb{R}$ *be a measurable set such that* $\mu(E)$ *exists and is non-zero and let* $N:\mathbb{R}\to\mathbb{R}$ *be an arbitrary measurable function. Then for any tree,* $\mathbb{T} \subseteq \mathbb{D}_{0,\alpha_f}$,

$$\sum_{s\in\mathbb{T}} |\langle\!\langle(\chi_{\omega_s^+}\circ N)\psi_{s,f},\chi_E\rangle\!\rangle\langle\!\langle f,\psi_{s,f}\rangle\!\rangle| \lesssim \frac{1}{\sqrt{\alpha_f}}|I_{\text{top}(\mathbb{T})}|\mathcal{E}(f;\mathbb{T})\mathcal{M}(E;\mathbb{T})\|f\|_{B^2}\mu(E).$$

Proofs of the mass, energy and tree lemmata will be provided in Sections 4.1, 4.2 and 4.3 respectively.

For the purpose of simplifying the notation used, for the remainder of this thesis, the objects α_f and $\mathbb{P}_f$ will simply be written as α and $\mathbb{P}$.

The proof of Theorem 3.5.2 can now be completed, assuming the three lemmata. The scheme of the proof is to decompose the collection $\mathbb{P}$ into certain well-chosen subcollections of trees using the mass and energy lemmata, allowing the desired estimate to be proved using the tree lemma. More specifically, $\mathbb{P}$ will be written as $\bigsqcup_{n=-\infty}^{n_0} \mathbb{P}_n$ for some $n_0 \in \mathbb{Z}$ where for each n,

$$\mathcal{M}(E;\mathbb{P}_n) \leqslant 2^{2n}, \qquad \mathcal{E}(f;\mathbb{P}_n) \leqslant \frac{1}{|\Lambda_f|^{\frac{1}{2}}} 2^n$$

and $\mathbb{P}_n$ is a union of trees $\mathbb{T}_{n_j}$ such that

$$\sum_j |I_{n_j}| \lesssim 2^{-2n}.$$

To begin this process, choose $n_0 \in \mathbb{Z}$ to be sufficiently large that $\mathcal{M}(E;\mathbb{P}) \leqslant 2^{2n_0}$ and $\mathcal{E}(f;\mathbb{P}) \leqslant \frac{1}{|\Lambda_f|^{\frac{1}{2}}} 2^{n_0}$. Note that both $\mathcal{M}(E;\mathbb{P})$ and $\mathcal{E}(f;\mathbb{P})$ will decrease as tiles are removed from $\mathbb{P}$. The mass and energy lemmata will be used repeatedly to select which tiles are removed into the sets $\mathbb{P}_n$ so that the desired mass and energy estimates hold.

Consider the following steps:

- Let $\mathbb{P}_{n_0} = \emptyset$.

- If $\mathcal{M}(E;\mathbb{P}) > 2^{2(n_0-1)}$, apply the mass lemma to $\mathbb{P}$ to obtain the collections $\mathbb{P}^{\text{heavy}}$ and $\mathbb{P}^{\text{light}}$. Augment $\mathbb{P}_{n_0}$ by $\mathbb{P}^{\text{heavy}}$ and observe that $\mathbb{P}^{\text{heavy}}$ is a union of trees $\mathbb{T}'_j$ such that

  $$\sum_j |I_{\text{top}(\mathbb{T}'_j)}| \lesssim (\mathcal{M}(E;\mathbb{P}))^{-1} \leqslant 2^{-2(n_0-1)} \approx 2^{-2n_0}.$$

  Replace $\mathbb{P}$ with $\mathbb{P}^{\text{light}}$.

- If $\mathcal{E}(f;\mathbb{P}) > \frac{1}{|\Lambda_f|^{\frac{1}{2}}} 2^{n_0-1}$, apply the energy lemma to $\mathbb{P}$ to obtain the collections $\mathbb{P}^{\text{low}}$ and $\mathbb{P}^{\text{high}}$. Augment $\mathbb{P}_{n_0}$ by $\mathbb{P}^{\text{high}}$ and observe that $\mathbb{P}^{\text{high}}$ is a union of trees $\mathbb{T}''_j$ such that

  $$\sum_j |I_{\text{top}(\mathbb{T}''_j)}| \lesssim \frac{1}{|\Lambda_f|} \left(\frac{1}{|\Lambda_f|^{\frac{1}{2}}} \mathcal{E}(f;\mathbb{P}) \right)^{-2} \lesssim 2^{-2(n_0-1)} \approx 2^{-2n_0}.$$

  Replace $\mathbb{P}$ with $\mathbb{P}^{\text{low}}$.

- Observe that the newly obtained $\mathbb{P}_{n_0}$ is a union of trees $\mathbb{T}_{n_{0_j}}$ such that

$$\sum_j |I_{\mathrm{top}(\mathbb{T}_{n_{0_j}})}| \lesssim 2^{-2n_0} + 2^{-2n_0} \approx 2^{-2n_0}.$$

The above steps provide the set $\mathbb{P}_{n_0}$. To obtain the full sequence $(\mathbb{P}_n)_{-\infty}^{n_0}$, iterate the above procedure on what remains in the collection $\mathbb{P}$ (to obtain $\mathbb{P}_{n_0-1}$, then $\mathbb{P}_{n_0-2}$ and so forth).

With the desired sequence of subcollections of $\mathbb{P}$ obtained, now observe that for any $s \in \mathbb{D}_{0,\alpha}$,

$$\begin{aligned}
\mathcal{M}(E;\{s\}) &= \mu(E)^{-1} \sup_{\substack{u\in\mathbb{D}_{0,\alpha}\\ s<u}} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E\cap N^{-1}(\omega_u)} \sum_{m\in\mathbb{Z}} \frac{|I_u|^{-1}\,dx}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_u)|}{|I_u|}\right)^{10}} \\
&\leqslant \mu(E)^{-1} \sup_{\substack{u\in\mathbb{D}_{0,\alpha}\\ s<u}} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{m\in\mathbb{Z}} \frac{|I_u|^{-1}\,dx}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_u)|}{|I_u|}\right)^{10}} \\
&= \mu(E)^{-1} \sup_{\substack{u\in\mathbb{D}_{0,\alpha}\\ s<u}} \alpha \int_0^{\frac{1}{\alpha}} \sum_{m\in\mathbb{Z}} \frac{|I_u|^{-1}\,dx}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_u)|}{|I_u|}\right)^{10}} \\
&= \mu(E)^{-1} \sup_{\substack{u\in\mathbb{D}_{0,\alpha}\\ s<u}} \alpha \int_{\mathbb{R}} \frac{|I_u|^{-1}\,dx}{\left(1+\frac{|x|}{|I_u|}\right)^{10}} \\
&= \mu(E)^{-1}\alpha \int_{\mathbb{R}} (1+|x|)^{-10}\,dx \\
&\leqslant \frac{\alpha}{\mu(E)}.
\end{aligned}$$

As such, the mass estimates on the collection $(\mathbb{P}_n)$ may be improved to

$$\mathcal{M}(E;\mathbb{P}_n) \leqslant \sqrt{\alpha}\min\left(\frac{2^{2n}}{\sqrt{\alpha}}, \frac{\sqrt{\alpha}}{\mu(E)}\right).$$

The proof of the main estimate (from Theorem 3.5.2) may now be completed as follows:

$$\begin{aligned}
&\sum_{s\in\mathbb{P}} |\langle\!\langle (\chi_{\omega_s^+}\circ N)\psi_{s,f}, \chi_E\rangle\!\rangle \langle\!\langle f, \psi_{s,f}\rangle\!\rangle| \\
\leqslant\ & \sum_{n=-\infty}^{n_0} \sum_j \sum_{s\in\mathbb{T}_{n_j}} |\langle\!\langle (\chi_{\omega_s^+}\circ N)\psi_{s,f}, \chi_E\rangle\!\rangle \langle\!\langle f, \psi_{s,f}\rangle\!\rangle|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{\sqrt{\alpha}}\mu(E)\|f\|_{B^2}\sum_{n=-\infty}^{n_0}\sum_j |I_{n_j}|\mathscr{E}(f;\mathbb{T}_{n_j})\mathscr{M}(E;\mathbb{T}_{n_j}) \text{ by the tree lemma}\\
&\leqslant \frac{1}{|\Lambda_f|^{\frac{1}{2}}}\mu(E)\|f\|_{B^2}\sum_{n=-\infty}^{n_0}2^n\min\left(\frac{2^{2n}}{\sqrt{\alpha}},\frac{\sqrt{\alpha}}{\mu(E)}\right)\sum_j |I_{n_j}| \text{ by the mass/energy estimates}\\
&\lesssim \frac{1}{|\Lambda_f|^{\frac{1}{2}}}\mu(E)^{\frac{1}{2}}\sum_{n=-\infty}^{n_0}2^n\min\left(\left(\frac{\mu(E)}{\alpha}\right)^{\frac{1}{2}}2^{2n},\left(\frac{\mu(E)}{\alpha}\right)^{-\frac{1}{2}}\right)2^{-2n}\\
&\leqslant \frac{1}{|\Lambda_f|^{\frac{1}{2}}}\mu(E)^{\frac{1}{2}}\|f\|_{B^2}\sum_{n\in\mathbb{Z}}\min\left(\left(\frac{\mu(E)}{\alpha}\right)^{\frac{1}{2}}2^{n},\left(\frac{\mu(E)}{\alpha}\right)^{-\frac{1}{2}}2^{-n}\right)\\
&\lesssim \frac{1}{|\Lambda_f|^{\frac{1}{2}}}\mu(E)^{\frac{1}{2}}\|f\|_{B^2}.
\end{aligned}
$$

This completes the proof of Theorem 3.5.2, given the three lemmata.

CHAPTER 4

THE THREE LEMMATA

This chapter will provide the proofs of the three lemmata stated in Section 3.5. The broad schemes of the proofs are based upon the analogous proofs by Lacey and Thiele in the context of the Carleson operator on $L^2(\mathbb{R})$, and as such, the reader may find it instructive to compare the steps presented herein with those followed in [92]. More detailed accounts of Lacey and Thiele's proofs are also provided in [90] and [67, Ch. 11].

4.1 The Mass Lemma

In this section, a proof of the mass lemma is provided. Recall that for a tile $s \in \mathbb{D}_{0,\alpha}$ and a collection $\mathbb{S} \subseteq \mathbb{D}_{0,\alpha}$, mass is defined in the following way:

$$\mathcal{M}(E;\{s\}) := \mu(E)^{-1} \sup_{\substack{u \in \mathbb{D}_{0,\alpha} \\ s < u}} \lim_{T \to \infty} \frac{1}{2T} \int_{[-T,T] \cap E \cap N^{-1}(\omega_u)} \sum_{m \in \mathbb{Z}} \frac{|I_u|^{-1}}{\left(1 + \frac{|x + \frac{m}{\alpha} - c(I_u)|}{|I_u|}\right)^{10}} \, dx,$$

$$\mathcal{M}(E;\mathbb{S}) := \sup_{s \in \mathbb{S}} \mathcal{M}(E;\{s\}).$$

This definition given, the mass lemma, as stated in Section 3.5, is as follows:

The Mass Lemma *Let f be a trigonometric polynomial and let $N : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function. Choose $E \subseteq \mathbb{R}$ to be a measurable set such that $\mu(E)$ exists and is positive* *and let $\mathbb{P}$ be an arbitrary finite collection of tiles from $\mathbb{D}_{0,\alpha}$. Then $\mathbb{P}$ can be written as $\mathbb{P}^{\text{light}} \sqcup \mathbb{P}^{\text{heavy}}$ where*

$$\mathscr{M}(E;\mathbb{P}^{\text{light}}) \leqslant \frac{1}{4}\mathscr{M}(E;\mathbb{P})$$

and $\mathbb{P}^{\text{heavy}}$ is a union of trees $\mathbb{T}_j$ with

$$\sum_j |I_j| \lesssim \mathscr{M}(E;\mathbb{P})^{-1}.$$

Proof The decomposition of $\mathbb{P}$ into the disjoint union of $\mathbb{P}^{\text{light}}$ and $\mathbb{P}^{\text{heavy}}$ is straightforward. To select $\mathbb{P}^{\text{heavy}}$, simply choose "heavy tiles" in the following way:

$$\mathbb{P}^{\text{heavy}} := \{s \in \mathbb{P} : \mathscr{M}(E;\{s\}) > \tfrac{1}{4}\mathscr{M}(E;\mathbb{P})\}.$$

It is then clear that $\mathbb{P}^{\text{light}}$, defined to be the collection of remaining tiles, satisfies $\mathscr{M}(E;\mathbb{P}^{\text{light}}) \leqslant \frac{1}{4}\mathscr{M}(E;\mathbb{P})$, so to prove the lemma, it suffices to show that $\mathbb{P}^{\text{heavy}}$ is a union of trees $\mathbb{T}_j$ such that $\sum_j |I_j| \lesssim \mathscr{M}(E;\mathbb{P})^{-1}$.

Observe that for each $s \in \mathbb{P}^{\text{heavy}}$, there exists $u_s \in \mathbb{D}_{0,\alpha}$ with $s < u_s$ such that

$$\mu(E)^{-1} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E\cap N^{-1}(\omega_{u_s})} \sum_{m\in\mathbb{Z}} \frac{|I_{u_s}|^{-1}}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_{u_s})|}{|I_{u_s}|}\right)^{10}}\,dx > \frac{1}{4}\mathscr{M}(E;\mathbb{P}).$$

The symbol $\mathbb{U}$ will be used to denote the collection of such tiles, that is to say

$$\mathbb{U} := \{u_s : s \in \mathbb{P}^{\text{heavy}}\}.$$

The collection $\mathbb{P}^{\text{heavy}}$ can be grouped into trees $\mathbb{T}_j$ by selecting the maximal elements (with respect to the partial order $<$) from $\mathbb{P}^{\text{heavy}}$ as the tops of these trees, recalling that distinct maximal tiles from a given collection are necessarily disjoint. Let $\mathbb{U}_{\max}$ be the collection of maximal tiles from $\mathbb{U}$. Note that for any $u \in \mathbb{U}_{\max}$, if there exist j and j' such that $\text{top}(\mathbb{T}_j) < u$

and $\operatorname{top}(\mathbb{T}_{j'}) < u$, it must be the case that $I_j \cap I_{j'} = \emptyset$. Indeed, if this were not the case, the disjointness of the tops of the trees would require that $\omega_j \cap \omega_{j'} = \emptyset$ which would contradict the fact that $\operatorname{top}(\mathbb{T}_j) < u$ and $\operatorname{top}(\mathbb{T}_{j'}) < u$. It follows from this observation that for any distinct j and j', either the time projections of the tops of $\mathbb{T}_j$ and $\mathbb{T}_{j'}$ are disjoint or they are contained within the time projections of distinct tiles from $\mathbb{U}_{\max}$ (or both), hence

$$\sum_j |I_j| \leqslant \sum_{u \in \mathbb{U}_{\max}} |I_u|.$$

As such, to prove the desired inequality, $\sum_j |I_j| \lesssim \mathscr{M}(E;\mathbb{P})^{-1}$, it may be assumed without loss of generality that the tops of the trees, $\mathbb{T}_j$, are precisely the elements of $\mathbb{U}_{\max}$. Given this assumption, for each j,

$$\mu(E)^{-1} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E\cap N^{-1}(\omega_j)} \sum_{m\in\mathbb{Z}} \frac{|I_j|^{-1}}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_j)|}{|I_j|}\right)^{10}}\, dx > \frac{1}{4}\mathscr{M}(E;\mathbb{P}).$$

Introducing a mild abuse of notation by temporarily defining $2^{-1} \star I_j := \emptyset$, each side of the above inequality may be decomposed as follows:

$$\begin{aligned}
&\mu(E)\frac{\mathscr{M}(E;\mathbb{P})}{8}|I_j|\sum_{k=0}^{\infty} 2^{-k} \\
<\ & \sum_{k=0}^{\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E\cap N^{-1}(\omega_j)} \sum_{\substack{m\in\mathbb{Z} \\ x+\frac{m}{\alpha}\in 2^k\star I_j \setminus 2^{k-1}\star I_j}} \left(1+\frac{|x+\frac{m}{\alpha}-c(I_j)|}{|I_j|}\right)^{-10} dx.
\end{aligned}$$

Note that the interchange of the sum in k and the integral can be justified by the monotone convergence theorem. To justify the interchange of the sum in k and the limit in T, by the dominated convergence theorem (formulated for infinite sums), it suffices to show that

$$\frac{1}{2T} \int_{[-T,T]\cap E\cap N^{-1}(\omega_j)} \sum_{\substack{m\in\mathbb{Z} \\ x+\frac{m}{\alpha}\in 2^k\star I_j \setminus 2^{k-1}\star I_j}} \left(1+\frac{|x+\frac{m}{\alpha}-c(I_j)|}{|I_j|}\right)^{-10} dx$$

is bounded uniformly in T by a quantity that is summable in k. To prove this, observe that

$$\begin{aligned}
&\sup_{T\in\mathbb{R}^+}\frac{1}{2T}\int_{[-T,T]\cap E\cap N^{-1}(\omega_j)}\sum_{\substack{m\in\mathbb{Z}\\ x+\frac{m}{\alpha}\in 2^k\star I_j\setminus 2^{k-1}\star I_j}}\left(1+\frac{|x+\frac{m}{\alpha}-c(I_j)|}{|I_j|}\right)^{-10}dx\\
\leqslant\ &\sup_{x\in\mathbb{R}}\sum_{\substack{m\in\mathbb{Z}\\ x+\frac{m}{\alpha}\in 2^k\star I_j\setminus 2^{k-1}\star I_j}}\left(1+\frac{|x+\frac{m}{\alpha}-c(I_j)|}{|I_j|}\right)^{-10}\\
\leqslant\ &2^{k-1}\left(1+\frac{2^{k-2}|I_j|}{|I_j|}\right)^{-10}
\end{aligned}$$

as the sum in m consists of at most 2^{k-1} terms and $\inf\limits_{z\in 2^k\star I_j\setminus 2^{k-1}\star I_j}|z-c(I_j)|=2^{k-2}|I_j|$.

As a consequence of this decomposition, for each j, there exists $k\in\mathbb{N}_0$ such that

$$\begin{aligned}
&\mu(E)\frac{\mathscr{M}(E;\mathbb{P})}{8}|I_j|2^{-k}\\
<\ &\lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap E\cap N^{-1}(\omega_j)}\sum_{\substack{m\in\mathbb{Z}\\ x+\frac{m}{\alpha}\in 2^k\star I_j\setminus 2^{k-1}\star I_j}}\left(1+\frac{|x+\frac{m}{\alpha}-c(I_j)|}{|I_j|}\right)^{-10}dx.
\end{aligned}$$

Denote by $\mathbb{U}_k$ the set of the tops of all trees from the collection $(\mathbb{T}_j)$ that satisfy the above estimate for each particular $k\in\mathbb{N}_0$. Observe that since there exists at least one such $k\in\mathbb{N}_0$ for each j, it follows that $\sum\limits_j|I_j|\leqslant\sum\limits_{k=0}^{\infty}\sum\limits_{u\in\mathbb{U}_k}|I_u|$.

For fixed $k\in\mathbb{N}_0$, successively select $u\in\mathbb{U}_k$ such that $|I_u|$ is as large as possible but so that $(2^k\star I_u)\times\omega_u$ is disjoint from $(2^k\star I_{u'})\times\omega_{u'}$ for previously selected $u'\in\mathbb{U}_k$. Once this process is exhausted, for any $u\in\mathbb{U}_k$, there exists a selected $u'\in\mathbb{U}_k$ such that $|I_u|\leqslant|I_{u'}|$ with the corresponding enlarged tiles intersecting. It follows that all tiles from $\mathbb{U}_k$ associated with the same selected tile u' have their time projections covered by $2^{k+2}\star I_{u'}$. Further, by maximality, since their frequency projections must overlap, their time projections must be disjoint. As such,

$$\sum_{u\in\mathbb{U}_k}|I_u|\leqslant 2^{k+2}\sum_{\substack{u'\in\mathbb{U}_k\\ u'\text{selected}}}|I_{u'}|,$$

so it follows that

$$\sum_{u \in \mathbb{U}_k} \mu(E) \frac{\mathcal{M}(E;\mathbb{P})}{8} |I_u| 2^{-k}$$
$$< \quad 2^{k+2} \sum_{\substack{u' \in \mathbb{U}_k \\ u' \text{selected}}} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E \cap N^{-1}(\omega_{u'})} \sum_{\substack{m\in\mathbb{Z} \\ x+\frac{m}{\alpha} \in 2^k \star I_{u'} \setminus 2^{k-1}\star I_{u'}}} \left(1 + \frac{|x+\frac{m}{\alpha} - c(I_{u'})|}{|I_{u'}|}\right)^{-10} dx.$$

For any $x \in \mathbb{R}$, the sum in m has finitely many terms. In particular,

$$\begin{aligned}
& |\{m \in \mathbb{Z} : x + \tfrac{m}{\alpha} \in 2^k \star I_{u'} \setminus 2^{k-1} \star I_{u'}\}| \\
\leqslant \quad & |\{m \in \mathbb{Z} : \tfrac{m}{\alpha} \in 2^k \star [0, \tfrac{1}{\alpha}]\}| \\
= \quad & |\{m \in \mathbb{Z} : m \in 2^k \star [0,1]\}| \\
\leqslant \quad & 2|\{m \in \mathbb{Z} : m \in [0, 2^k]\}| \\
\leqslant \quad & 2^{k+2}.
\end{aligned}$$

With this estimate on the number of terms in the sum in m established, making a straightforward ℓ^∞ estimate, it can be seen that

$$\mu(E) \frac{\mathcal{M}(E;\mathbb{P})}{8} 2^{-k} \sum_{u \in \mathbb{U}_k} |I_u|$$
$$< \quad 2^{k+2} \sum_{\substack{u' \in \mathbb{U}_k \\ u' \text{selected}}} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E \cap N^{-1}(\omega_{u'})} 2^{k+2} \max_{m\in\mathbb{Z}} \left[\left(1 + \frac{|x+\frac{m}{\alpha} - c(I_{u'})|}{|I_{u'}|}\right)^{-10} \right.$$
$$\left. \times \chi_{2^k \star I_{u'} \setminus 2^{k-1} \star I_{u'}} \left(x + \frac{m}{\alpha}\right) \right] dx.$$

Observe that for $x + \dfrac{m}{\alpha} \in 2^k \star I_{u'} \setminus 2^{k-1} \star I_{u'}$,

$$1 + \frac{|x + \frac{m}{\alpha} - c(I_{u'})|}{|I_{u'}|} \geqslant \frac{4}{5}(1 + 2^{k-2}).$$

It hence follows that

$$\begin{aligned}
&\mu(E)\frac{\mathcal{M}(E;\mathbb{P})}{8}2^{-k}\sum_{u\in\mathbb{U}_k}|I_u| \\
\lesssim\ & 2^k \sum_{\substack{u'\in\mathbb{U}_k \\ u'\text{selected}}} \lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap E\cap N^{-1}(\omega_{u'})} 2^{-9k}\max_{m\in\mathbb{Z}}\chi_{2^k\star I_{u'}\backslash 2^{k-1}\star I_{u'}}(x+\tfrac{m}{\alpha})\,dx \\
=\ & 2^{-8k}\sum_{\substack{u'\in\mathbb{U}_k \\ u'\text{selected}}} \mu\Big(E\cap N^{-1}(\omega_{u'})\cap\bigcup_{m\in\mathbb{Z}}\big((2^k\star I_{u'}\backslash 2^{k-1}\star I_{u'})+\tfrac{m}{\alpha}\big)\Big).
\end{aligned}$$

Now, as the sets $2^k\star I_{u'}\times\omega_{u'}$ are disjoint for selected tiles u' from $\mathbb{U}_k$, it can be seen that for any collection of such tiles, $\mathbb{V}$, such that the $N^{-1}(\omega_{u'})$ overlap, the $2^k\star I_{u'}$ must be pairwise disjoint. Observing that

$$\sum_{u'\in\mathbb{V}}|I_{u'}|\lesssim\max\Big(\sum_{\substack{u'\in\mathbb{V}\\ I_{u'}\subseteq[0,\frac{1}{2\alpha}]}}|I_{u'}|,\ \sum_{\substack{u'\in\mathbb{V}\\ I_{u'}\subseteq[\frac{1}{2\alpha},\frac{1}{\alpha}]}}|I_{u'}|\Big),$$

the collection $\mathbb{V}$ can be restricted to those u' such that all the $I_{u'}$ are contained in either $[0,\frac{1}{2\alpha}]$ or $[\frac{1}{2\alpha},\frac{1}{\alpha}]$.* It is claimed that under this assumption, if there is more than one tile remaining in $\mathbb{V}$ then the $2^k\star I_{u'}$ are not only disjoint but must also all be contained within an interval of width $\frac{1}{\alpha}$. Indeed, this follows from the following elementary result:

Proposition *A collection of non-overlapping intervals with centres all contained in an interval of width l must be entirely contained within an interval of width $2l$.*†

From this observation, it can be assumed without loss of generality that the sets

$$N^{-1}(\omega_{u'})\cap\bigcup_{m\in\mathbb{Z}}\big((2^k\star I_{u'}\backslash 2^{k-1}\star I_{u'})+\tfrac{m}{\alpha}\big)$$

*Since the hypothesis that $\mathbb{V}$ contains tiles u' with $N^{-1}(\omega_{u'})$ overlapping necessarily means that $\mathbb{V}$ contains more than one tile and since the $I_{u'}$ are disjoint, having a tile u' in $\mathbb{V}$ with $I_{u'}=[0,\frac{1}{\alpha})$ is not possible.

†To prove this proposition, it suffices to consider the case where the collection consists of only two intervals.

are pairwise disjoint across the collection of selected tiles from $\mathbb{U}_k$ and hence

$$\begin{aligned}
&\mu(E)\frac{\mathcal{M}(E;\mathbb{P})}{8}2^{-k}\sum_{u\in\mathbb{U}_k}|I_u| \\
\lesssim\;& 2^{-8k}\sum_{\substack{u'\in\mathbb{U}_k\\ u'\text{selected}}}\mu\Big(E\cap N^{-1}(\omega_{u'})\cap\bigcup_{m\in\mathbb{Z}}\big((2^k\star I_{u'}\backslash 2^{k-1}\star I_{u'})+\tfrac{m}{\alpha}\big)\Big) \\
\leqslant\;& 2^{-8k}\mu(E).
\end{aligned}$$

Recalling that $\sum_j |I_j|\leqslant\sum_{k=0}^{\infty}\sum_{u\in\mathbb{U}_k}|I_u|$, it follows that

$$\begin{aligned}
\sum_j |I_j| &\lesssim \sum_{k=0}^{\infty}2^{-7k}\mathcal{M}(E;\mathbb{P})^{-1} \\
&\lesssim \mathcal{M}(E;\mathbb{P})^{-1},
\end{aligned}$$

which completes the proof of the mass lemma. □

4.2 The Energy Lemma

This section will provide a proof of the energy lemma from Section 3.5. Recall that the energy of the function f with respect to a collection of tiles, $\mathbb{S}\subseteq\mathbb{D}_{0,\alpha}$, is given by the following expression:

$$\mathcal{E}(f;\mathbb{S})=\frac{1}{\|f\|_{B^2}}\sup_{\substack{\mathbb{T}\text{ a +tree}\\ \mathbb{T}\subseteq\mathbb{S}}}\left(\frac{1}{|I_{\text{top}(\mathbb{T})}|}\sum_{s\in\mathbb{T}}|\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|^2\right)^{\frac{1}{2}}.$$

The energy lemma is as follows:

The Energy Lemma *Let f be a trigonometric polynomial and let $\mathbb{P}$ be an arbitrary finite collection of tiles from $\mathbb{D}_{0,\alpha}$. Then $\mathbb{P}$ can be written as $\mathbb{P}^{\text{low}}\sqcup\mathbb{P}^{\text{high}}$ where*

$$\mathcal{E}(f;\mathbb{P}^{\text{low}})\leqslant\frac{1}{2}\mathcal{E}(f;\mathbb{P})$$

and $\mathbb{P}^{\text{high}}$ *is a union of trees* $\mathbb{T}_j$ *such that*

$$\sum_j |I_j| \lesssim \frac{1}{|\Lambda_f|}\mathscr{E}(f;\mathbb{P})^{-2}.$$

Proof Firstly, for any +tree, $\mathbb{T}$, define the quantity $\Delta(\mathbb{T}) := \frac{1}{\|f\|_{B^2}}\left(\frac{1}{|I_{\text{top}(\mathbb{T})}|}\sum_{s\in\mathbb{T}}|\langle f,\psi_{s,f}\rangle|^2\right)^{\frac{1}{2}}$.

The process of partitioning $\mathbb{P}$ into $\mathbb{P}^{\text{low}}$ and $\mathbb{P}^{\text{high}}$ is essentially as straightforward as the analogous step from the proof of the mass lemma. However, a particular selection algorithm will be used to select entire trees from $\mathbb{P}$ to form the collection $\mathbb{P}^{\text{high}}$. Specifically, the "high energy" tiles should be selected by repeating the following procedure for increasing $j \in \mathbb{N}$ until it terminates:

- Choose the +tree $\mathbb{T}_j+ \subseteq \mathbb{P}$ with minimal $c(\omega_{\text{top}(\mathbb{T}_j+)})$ such that $\Delta(\mathbb{T}_j+) > \frac{1}{2}\mathscr{E}(f;\mathbb{P})$.

- Let $\mathbb{T}_j \subseteq \mathbb{P}$ be the largest tree possible (in terms of its cardinality) with the same top as $\mathbb{T}_j+$ and remove it from $\mathbb{P}$.

Define $\mathbb{P}^{\text{low}}$ to be the remaining tiles in $\mathbb{P}$. Since it is clear that $\mathscr{E}(f;\mathbb{P}^{\text{low}}) \leqslant \frac{1}{2}\mathscr{E}(f;\mathbb{P})$, the proof will now be complete once it has been shown that $\sum_j |I_j| \lesssim \mathscr{E}(f;\mathbb{P})^{-2}$.

Using the fact that $\Delta(\mathbb{T}_j+) > \frac{1}{2}\mathscr{E}(f;\mathbb{P})$ for each j,

$$\begin{aligned}
\frac{1}{4}\mathscr{E}(f;\mathbb{P})^2\sum_j|I_j| &< \frac{1}{\|f\|_{B^2}^2}\sum_j\sum_{s\in\mathbb{T}_j+}|\langle f,\psi_{s,f}\rangle|^2 \\
&= \frac{1}{\|f\|_{B^2}^2}\sum_j\sum_{s\in\mathbb{T}_j+}\langle f,\psi_{s,f}\rangle\overline{\langle f,\psi_{s,f}\rangle} \\
&= \frac{1}{\|f\|_{B^2}^2}\left\langle f, \sum_j\sum_{s\in\mathbb{T}_j+}\langle f,\psi_{s,f}\rangle\psi_{s,f}\right\rangle \\
&\leqslant \frac{1}{\|f\|_{B^2}}\left\|\sum_j\sum_{s\in\mathbb{T}_j+}\langle f,\psi_{s,f}\rangle\psi_{s,f}\right\|_{B^2}.
\end{aligned}$$

It thus suffices to show that

$$\left\|\sum_j\sum_{s\in\mathbb{T}_j+}\langle f,\psi_{s,f}\rangle\psi_{s,f}\right\|_{B^2}^2 \lesssim \frac{1}{|\Lambda_f|}\|f\|_{B^2}^2\mathscr{E}(f;\mathbb{P})^2\sum_j|I_j|.$$

Define $\mathbb{U} := \bigcup_j \mathbb{T}_j +$ and note that

$$\begin{aligned}
&\left\| \sum_{s\in\mathbb{U}} \langle f, \psi_{s,f} \rangle \psi_{s,f} \right\|_{B^2}^2 \\
= \;& \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{s,s'\in\mathbb{U}} \langle f, \psi_{s,f} \rangle \psi_{s,f} \overline{\langle f, \psi_{s',f} \rangle \psi_{s',f}} \\
\leqslant \;& \sum_{s,s'\in\mathbb{U}} |\langle f, \psi_{s,f} \rangle \langle \psi_{s,f}, \psi_{s',f} \rangle \langle f, \psi_{s',f} \rangle|.
\end{aligned}$$

Now, by orthogonality on the frequency side (Proposition 2.3.3), it can be seen, given the dyadic sizing of the tiles, that unless $\omega_s^- \subseteq \omega_{s'}^-$ or vice versa, it must be the case that

$$\langle \psi_{s,f}, \psi_{s',f} \rangle = 0.$$

Using this observation and a symmetry argument,

$$\begin{aligned}
&\left\| \sum_j \sum_{s\in\mathbb{T}_j+} \langle f, \psi_{s,f} \rangle \psi_{s,f} \right\|_{B^2}^2 \\
\leqslant \;& \sum_{\substack{s,s'\in\mathbb{U} \\ \omega_s = \omega_{s'}}} |\langle f, \psi_{s,f} \rangle \langle \psi_{s,f}, \psi_{s',f} \rangle \langle f, \psi_{s',f} \rangle| + 2 \sum_{\substack{s,s'\in\mathbb{U} \\ \omega_s \subseteq \omega_{s'}^-}} |\langle f, \psi_{s,f} \rangle \langle \psi_{s,f}, \psi_{s',f} \rangle \langle f, \psi_{s',f} \rangle|.
\end{aligned}$$

The two terms in the above expression will simply be referred to as the “first term” and the “second term” and they will be considered separately.

For the first term, recall that by Proposition 2.3.3, for tiles $s,\, s' \in \mathbb{D}_{0,\alpha}$ with $|I_s| \geqslant |I_{s'}|$,

$$|\langle \psi_{s,f}, \psi_{s',f} \rangle| \lesssim \frac{1}{|\Lambda_f|} \left(\frac{|I_s|}{|I_{s'}|} \right)^{\frac{1}{2}} \sum_{l\in\mathbb{Z}} \int_{I_{s'}+\frac{l}{\alpha}} \frac{|I_s|^{-1}\, dx}{\left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^{10}}.$$

Note that for any distinct tiles, $s,\, s' \in \mathbb{D}_{0,\alpha}$, with $\omega_s = \omega_{s'}$ it is necessarily the case that $I_s \cap I_{s'} = \emptyset$. As such, majorising $|\langle f, \psi_{s,f} \rangle|$ and $|\langle f, \psi_{s',f} \rangle|$ by the larger of the two, appealing to symmetry and using the above estimate, it can be seen that

$$
\begin{aligned}
& \sum_{\substack{s,s'\in\mathbb{U}\\ \omega_s=\omega_{s'}}} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\langle\!\langle \psi_{s,f},\psi_{s',f}\rangle\!\rangle\langle\!\langle f,\psi_{s',f}\rangle\!\rangle| \\
\leqslant\ & \sum_{s\in\mathbb{U}} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|^2 \sum_{\substack{s'\in\mathbb{U}\\ \omega_s=\omega_{s'}}} |\langle\!\langle \psi_{s,f},\psi_{s',f}\rangle\!\rangle| \\
\lesssim\ & \frac{1}{|\Lambda_f|}\sum_{s\in\mathbb{U}} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|^2 \sum_{\substack{s'\in\mathbb{U}\\ \omega_s=\omega_{s'}}} \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \sum_{l\in\mathbb{Z}} \int_{I_{s'}+\frac{l}{a}} \frac{|I_s|^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^{10}} \\
\lesssim\ & \frac{1}{|\Lambda_f|}\sum_{j}\sum_{s\in\mathbb{T}_j+} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|^2 \\
\leqslant\ & \frac{1}{|\Lambda_f|}\sum_{j} |I_j|\mathscr{E}(f;\mathbb{P})^2\|f\|_{B^2}^2.
\end{aligned}
$$

This gives the required estimate on the “first term”.

To prove the desired estimate for the “second term”, it will first be shown that there are “strong disjointness” properties that hold for the time projections of the tiles under consideration. The first such claimed property is that for $s\in\mathbb{T}_j+$ and $s'\in\mathbb{T}_k+$ such that $\omega_s\subseteq\omega_{s'}^-$, it is the case that $I_{s'}\cap I_j=\emptyset$. It can thus be said not only that the time projection of s' is disjoint from that of s, but also that it is disjoint from the time projection of the entire tree that s belongs to. Note that by the hypothesis that $\omega_s\subseteq\omega_{s'}^-$, the $+$trees here are necessarily distinct. The second strong disjointness property states that, given an additional tile, $s''\in\mathbb{T}_l+$ with $s''\neq s'$ and $\omega_s\subseteq\omega_{s''}^-$, it is necessarily the case that $I_{s'}\cap I_{s''}=\emptyset$.

To prove the first of these properties, observe that $c(\omega_j)\in\omega_s\subseteq\omega_{s'}^-$. As $\mathbb{T}_k+$ is a $+$tree, $c(\omega_k)\in\omega_{s'}^+$ and hence $c(\omega_j)<c(\omega_k)$. Given the algorithm used to select the $+$trees, it must thus be the case that $\mathbb{T}_j+$ was selected before $\mathbb{T}_k+$. Given that $|I_{s'}|<|I_j|$, it follows that if $I_{s'}\cap I_j$ were not empty, the tile s' would have qualified to be a member of $\mathbb{T}_j$, which would prohibit it from being in $\mathbb{T}_k+$. The first property hence follows.

The second strong disjointness property is a corollary of the first. Note that $\omega_{s'}^-$ and $\omega_{s''}^-$ have non-empty intersection (as they both contain ω_s). If $\omega_{s'}^-=\omega_{s''}^-$ then $I_{s'}$ and $I_{s''}$ must be disjoint by the assumption that $s'' \neq s'$. Assuming that this is not the case, either $\omega_{s'} \subseteq \omega_{s''}^-$ or $\omega_{s''} \subseteq \omega_{s'}^-$. The first strong disjointness property can be applied to both cases, providing that either $I_{s''} \cap I_k = \emptyset$ or $I_{s'} \cap I_l = \emptyset$. The truth of either of these statements requires that $I_{s'} \cap I_{s''} = \emptyset$ and so the second property is proved.

With these strong disjointness properties established, now consider that

$$\begin{aligned}
&\sum_j \sum_{s \in \mathbb{T}_j+} |\langle\!\langle f, \psi_{s,f} \rangle\!\rangle| \sum_{\substack{s' \in \mathbb{U} \\ \omega_s \subseteq \omega_{s'}^-}} |\langle\!\langle f, \psi_{s',f} \rangle\!\rangle \langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| \\
\leqslant\ & \sum_j \Big(\sum_{s \in \mathbb{T}_j+} |\langle\!\langle f, \psi_{s,f} \rangle\!\rangle|^2 \Big)^{\frac{1}{2}} \Bigg(\sum_{s \in \mathbb{T}_j+} \Big(\sum_{\substack{s' \in \mathbb{U} \\ \omega_s \subseteq \omega_{s'}^-}} |\langle\!\langle f, \psi_{s',f} \rangle\!\rangle \langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| \Big)^2 \Bigg)^{\frac{1}{2}} \\
\leqslant\ & \sum_j |I_j|^{\frac{1}{2}} \mathscr{E}(f; \mathbb{P}) \|f\|_{B^2} \Bigg(\sum_{s \in \mathbb{T}_j+} \Big(\sum_{\substack{s' \in \mathbb{U} \\ \omega_s \subseteq \omega_{s'}^-}} |\langle\!\langle f, \psi_{s',f} \rangle\!\rangle \langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| \Big)^2 \Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Now, as the singleton $\{s'\}$ is a +tree, $|\langle\!\langle f, \psi_{s',f} \rangle\!\rangle| \leqslant |I_{s'}|^{\frac{1}{2}} \mathscr{E}(f; \mathbb{P}) \|f\|_{B^2}$, so the above expression is bounded by

$$\mathscr{E}(f; \mathbb{P})^2 \|f\|_{B^2}^2 \sum_j |I_j|^{\frac{1}{2}} \Bigg(\sum_{s \in \mathbb{T}_j+} \Big(\sum_{\substack{s' \in \mathbb{U} \\ \omega_s \subseteq \omega_{s'}^-}} |I_{s'}|^{\frac{1}{2}} |\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| \Big)^2 \Bigg)^{\frac{1}{2}}.$$

Considering the inner two sums only and using the estimate on the inner product from Proposition 2.3.3 again,

$$\begin{aligned}
&\sum_{s \in \mathbb{T}_j+} \Big(\sum_{\substack{s' \in \mathbb{U} \\ \omega_s \subseteq \omega_{s'}^-}} |I_{s'}|^{\frac{1}{2}} |\langle\!\langle \psi_{s,f}, \psi_{s',f} \rangle\!\rangle| \Big)^2 \\
\lesssim\ & \frac{1}{|\Lambda_f|^2} \sum_{s \in \mathbb{T}_j+} \left(\sum_{\substack{s' \in \mathbb{U} \\ \omega_s \subseteq \omega_{s'}^-}} |I_s|^{\frac{1}{2}} \sum_{l \in \mathbb{Z}} \int_{I_{s'} + \frac{l}{\alpha}} \frac{|I_s|^{-1}\, dx}{(1 + \frac{|x - c(I_s)|}{|I_s|})^{10}} \right)^2 \\
\leqslant\ & \frac{1}{|\Lambda_f|^2} \sum_{s \in \mathbb{T}_j+} |I_s| \left(\sum_{l \in \mathbb{Z}} \int_{([0, \frac{1}{\alpha}) \setminus I_j) + \frac{l}{\alpha}} \frac{|I_s|^{-1}\, dx}{(1 + \frac{|x - c(I_s)|}{|I_s|})^{10}} \right)^2 \text{ by strong disjointness}
\end{aligned}$$

$$\lesssim \frac{1}{|\Lambda_f|^2}\sum_{s\in\mathbb{T}_j+}|I_s|\sum_{l\in\mathbb{Z}}\int_{([0,\frac{1}{\alpha})\setminus I_j)+\frac{l}{\alpha}}\frac{|I_s|^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^{10}}.$$

To complete the proof of the energy lemma, it suffices to show that this quantity is bounded by a constant multiple of $\frac{1}{|\Lambda_f|^2}|I_j|$. Noting that the time projections of different tiles from $\mathbb{T}_j+$ at any particular scale are disjoint and decomposing the above quantity, it can be seen that

$$\begin{aligned}
&\frac{1}{|\Lambda_f|^2}\sum_{s\in\mathbb{T}_j+}|I_s|\sum_{l\in\mathbb{Z}}\int_{([0,\frac{1}{\alpha})\setminus I_j)+\frac{l}{\alpha}}\frac{|I_s|^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^{10}}\\
= {}& \frac{1}{|\Lambda_f|^2}\sum_{k=0}^{\infty}\sum_{\substack{s\in\mathbb{T}_j+\\ |I_s|=2^{-k}|I_j|}}2^{-k}|I_j|\sum_{l\in\mathbb{Z}}\int_{([0,\frac{1}{\alpha})\setminus I_j)+\frac{l}{\alpha}}\frac{(2^{-k}|I_j|)^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{2^{-k}|I_j|})^{10}}\\
\lesssim {}& \frac{1}{|\Lambda_f|^2}\sum_{k=0}^{\infty}\sum_{\substack{s\in\mathbb{T}_j+\\ |I_s|=2^{-k}|I_j|}}2^{-k}|I_j|\sum_{l\in\mathbb{Z}}\int_{I_s}\int_{([0,\frac{1}{\alpha})\setminus I_j)+\frac{l}{\alpha}}\frac{(2^{-k}|I_j|)^{-2}\,dx\,dy}{(1+\frac{|x-y|}{2^{-k}|I_j|})^{10}}\\
\leqslant {}& \frac{1}{|\Lambda_f|^2}\sum_{k=0}^{\infty}2^{-k}|I_j|\sum_{l\in\mathbb{Z}}\int_{I_j}\int_{([0,\frac{1}{\alpha})\setminus I_j)+\frac{l}{\alpha}}\frac{(2^{-k}|I_j|)^{-2}\,dx\,dy}{(1+\frac{|x-y|}{2^{-k}|I_j|})^{10}}\\
\lesssim {}& \frac{1}{|\Lambda_f|^2}|I_j|.
\end{aligned}$$

This completes the bound on the "second term" and thus completes the proof of the energy lemma. □

4.3 The Tree Lemma

This section will provide a proof of the tree lemma as stated in Section 3.5:

The Tree Lemma *Let f be a trigonometric polynomial, let $E\subseteq\mathbb{R}$ be a measurable set such that $\mu(E)$ exists and is non-zero and let $N:\mathbb{R}\to\mathbb{R}$ be an arbitrary measurable function. Then for any tree, $\mathbb{T}\subseteq\mathbb{D}_{0,\alpha}$,*

$$\sum_{s\in\mathbb{T}}|\langle\!\langle(\chi_{\omega_s^+}\circ N)\psi_{s,f},\chi_E\rangle\!\rangle\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|\lesssim\frac{1}{\sqrt{\alpha}}|I_{\mathrm{top}(\mathbb{T})}|\mathscr{E}(f;\mathbb{T})\mathscr{M}(E;\mathbb{T})\|f\|_{B^2}\mu(E).$$

Proof To begin with, by letting ε_s be an appropriate phase factor for each $s \in \mathbb{T}$, the left hand side of the desired inequality in the statement of the lemma can be bounded as follows:

$$\begin{aligned}
&\sum_{s\in\mathbb{T}} |\langle (\chi_{\omega_s^+} \circ N)\psi_{s,f}, \chi_E\rangle \langle f, \psi_{s,f}\rangle| \\
= &\sum_{s\in\mathbb{T}} \varepsilon_s \langle (\chi_{\omega_s^+} \circ N)\psi_{s,f}, \chi_E\rangle \langle f, \psi_{s,f}\rangle \\
= &\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{s\in\mathbb{T}} \varepsilon_s \langle f, \psi_{s,f}\rangle \chi_{E\cap N^{-1}(\omega_s^+)}(x)\psi_{s,f}(x)\, dx \\
\leqslant &\Big\| \sum_{s\in\mathbb{T}} \varepsilon_s \langle f, \psi_{s,f}\rangle \chi_{E\cap N^{-1}(\omega_s^+)} \psi_{s,f} \Big\|_{B^1}.
\end{aligned}$$

Throughout the remainder of this section, the term "$\frac{1}{\alpha}$-dyadic intervals" will be used to refer to the intervals of the form $[\frac{l}{\alpha}2^k, \frac{l+1}{\alpha}2^k)$ for $l,\, k \in \mathbb{Z}$. The time window, $[0, \frac{1}{\alpha})$, will be partitioned into maximal $\frac{1}{\alpha}$-dyadic intervals J such that $I_s \nsubseteq 3 \star J$ for all $s \in \mathbb{T}$. The collection of these intervals will be denoted by $\mathscr{J}$. For each $J \in \mathscr{J}$, $\widetilde{J}$ will be used to denote its $\frac{1}{\alpha}$-periodisation, that is $\widetilde{J} := \bigcup_{l\in\mathbb{Z}} (J + \frac{l}{\alpha})$.

This definition given, the left hand side of the desired inequality is seen to be bounded by the following decomposed expression:

$$\begin{aligned}
\sum_{s\in\mathbb{T}} |\langle (\chi_{\omega_s^+} \circ N)\psi_{s,f}, \chi_E\rangle \langle f, \psi_{s,f}\rangle| \quad\leqslant\quad &\sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T} \\ |I_s|\leqslant 2|J|}} \|\langle f, \psi_{s,f}\rangle \chi_{E\cap N^{-1}(\omega_s^+)} \psi_{s,f} \chi_{\widetilde{J}}\|_{B^1} \\
&+ \sum_{J\in\mathscr{J}} \Big\| \Big(\sum_{\substack{s\in\mathbb{T} \\ |I_s|>2|J|}} \varepsilon_s \langle f, \psi_{s,f}\rangle \chi_{E\cap N^{-1}(\omega_s^+)} \psi_{s,f} \Big) \chi_{\widetilde{J}} \Big\|_{B^1}.
\end{aligned}$$

For each $J \in \mathscr{J}$, this decomposition breaks up the tree into tiles with time projections that J stays away from and tiles with time projections that J is near to, loosely speaking. As such, these terms will be referred to as the "away" term and the "near" term and it will suffice to bound them by the right hand side of the desired inequality independently. Since the functions $\psi_{s,f}$ are "well localised" to the time projections, I_s, the "away" term is essentially a "Schwartz tails" term, whilst the "near" term is the more important and more difficult part of the estimate.

To deal with the "away" term, recall the estimate

$$|\psi_{s,f}(x)| \lesssim \frac{1}{\sqrt{\alpha}}|I_s|^{-\frac{1}{2}} \sum_{m\in\mathbb{Z}} \left(1+\frac{|x+\frac{m}{\alpha}-c(I_s)|}{|I_s|}\right)^{-20}$$

from Proposition 2.3.3 and the trivial energy estimate $|\langle\!\langle f,\psi_{s,f}\rangle\!\rangle| \leqslant \|f\|_{B^2}\mathscr{E}(f;\mathbb{T})|I_s|^{\frac{1}{2}}$, which follows as the singleton $\{s\}$ is a +tree. These given, it can be seen that

$$\begin{aligned}
&\sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T}\\ |I_s|\leqslant 2|J|}} \|\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}\chi_{\tilde{J}\cap E\cap N^{-1}(\omega_s^+)}\|_{B^1}\\
\lesssim\;& \sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T}\\ |I_s|\leqslant 2|J|}} \|f\|_{B^2}\mathscr{E}(f;\mathbb{T})|I_s|\frac{1}{\sqrt{\alpha}}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{m\in\mathbb{Z}}\frac{|I_s|^{-1}}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_s)|}{|I_s|}\right)^{20}}\chi_{\tilde{J}\cap E\cap N^{-1}(\omega_s^+)}(x)\,dx\\
\leqslant\;& \sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T}\\ |I_s|\leqslant 2|J|}} \|f\|_{B^2}\mathscr{E}(f;\mathbb{T})|I_s|\frac{1}{\sqrt{\alpha}}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{m\in\mathbb{Z}}\frac{|I_s|^{-1}\chi_{\tilde{J}\cap E\cap N^{-1}(\omega_s^+)}(x)}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_s)|}{|I_s|}\right)^{10}}\,dx\right]\\
&\quad\times\sup_{\substack{x\in\tilde{J}\\ m\in\mathbb{Z}}}\left(1+\frac{|x+\frac{m}{\alpha}-c(I_s)|}{|I_s|}\right)^{-10}\\
\leqslant\;& \frac{1}{\sqrt{\alpha}}\mathscr{E}(f;\mathbb{T})\mathscr{M}(E;\mathbb{T})\|f\|_{B^2}\mu(E)\sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T}\\ |I_s|\leqslant 2|J|}}|I_s|\sup_{x\in\tilde{J}}\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{-10}.
\end{aligned}$$

To obtain the required bound, it thus suffices to prove that

$$\sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T}\\ |I_s|\leqslant 2|J|}}|I_s|\sup_{x\in\tilde{J}}\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{-10} \lesssim |I_{\mathrm{top}(\mathbb{T})}|.$$

The supremum here may essentially be taken over just J, though the full details are slightly technical and will be postponed until the end of this section (see page 121). Making this assumption and breaking the sum in s up into separate scales,

$$\begin{aligned}
\sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T}\\ |I_s|\leqslant 2|J|}}|I_s|\sup_{x\in\tilde{J}}\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{-10} &= \sum_{J\in\mathscr{J}} \sum_{\substack{s\in\mathbb{T}\\ |I_s|\leqslant 2|J|}}|I_s|\sup_{x\in J}\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{-10}\\
&\leqslant \sum_{J\in\mathscr{J}}\sum_{k=-\infty}^{\log_2 2\alpha|J|}\frac{2^k}{\alpha}\sum_{\substack{s\in\mathbb{T}\\ |I_s|=\frac{2^k}{\alpha}}}\left(1+\frac{\mathrm{dist}(J,I_s)}{\alpha^{-1}2^k}\right)^{-10}.
\end{aligned}$$

Now, $\alpha 2^{-k} \operatorname{dist}(J, I_s) \geqslant |I_{\mathrm{top}(\mathbb{T})}|^{-1} \operatorname{dist}(J, I_{\mathrm{top}(\mathbb{T})})$. Noting that tiles at a fixed scale are pairwise disjoint,

$$\sum_{\substack{s\in\mathbb{T}\\ |I_s|=\frac{2^k}{\alpha}}} \left(1+\frac{\operatorname{dist}(J,I_s)}{\alpha^{-1}2^k}\right)^{-5} \lesssim 1,$$

hence the above can be bounded by a constant multiple of

$$\sum_{J\in\mathscr{J}} \sum_{k=-\infty}^{\log_2 2\alpha|J|} \frac{2^k}{\alpha(1+\frac{\operatorname{dist}(J,I_{\mathrm{top}(\mathbb{T})})}{|I_{\mathrm{top}(\mathbb{T})}|})^5} \lesssim \sum_{J\in\mathscr{J}} \frac{|J|}{(1+\frac{\operatorname{dist}(J,I_{\mathrm{top}(\mathbb{T})})}{|I_{\mathrm{top}(\mathbb{T})}|})^5} \lesssim \sum_{J\in\mathscr{J}} \frac{|J|}{(1+\frac{\frac{3}{2}|I_{\mathrm{top}(\mathbb{T})}|+2\operatorname{dist}(J,I_{\mathrm{top}(\mathbb{T})})}{|I_{\mathrm{top}(\mathbb{T})}|})^5}.$$

For any $J\in\mathscr{J}$, if $x\in J$ then $|x-c(I_{\mathrm{top}(\mathbb{T})})| \leqslant |J| + \operatorname{dist}(J, I_{\mathrm{top}(\mathbb{T})}) + \frac{1}{2}|I_{\mathrm{top}(\mathbb{T})}|$. Further, given that $I_{\mathrm{top}(\mathbb{T})}$ and J are $\frac{1}{\alpha}$-dyadic intervals as well as the fact that $I_{\mathrm{top}(\mathbb{T})} \nsubseteq 3\star J$, either $J\subseteq I_{\mathrm{top}(\mathbb{T})}$ or $J\cap I_{\mathrm{top}(\mathbb{T})}=\emptyset$. In the first case, $|J|\leqslant |I_{\mathrm{top}(\mathbb{T})}|$. In the second case, if $|J|>|I_{\mathrm{top}(\mathbb{T})}|$ then $|J|\leqslant \operatorname{dist}(J, I_{\mathrm{top}(\mathbb{T})})$. In either case, it follows that $|J|\leqslant \operatorname{dist}(J, I_{\mathrm{top}(\mathbb{T})}) + |I_{\mathrm{top}(\mathbb{T})}|$. Consequently, $|x-c(I_{\mathrm{top}(\mathbb{T})})| \leqslant \frac{3}{2}|I_{\mathrm{top}(\mathbb{T})}| + 2\operatorname{dist}(J, I_{\mathrm{top}(\mathbb{T})})$, so the previous expression is bounded by a constant multiple of

$$\sum_{J\in\mathscr{J}} \int_J \left(1+\frac{|x-c(I_{\mathrm{top}(\mathbb{T})})|}{|I_{\mathrm{top}(\mathbb{T})}|}\right)^{-5} dx \lesssim |I_{\mathrm{top}(\mathbb{T})}|.$$

This completes the desired bound on the “away” term.

To consider the “near” term, note first that for any $J\in\mathscr{J}$ such that there exists $s\in\mathbb{T}$ with $2|J|<|I_s|$, it is necessarily the case that $J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}$ since $2|J|<|I_{\mathrm{top}(\mathbb{T})}|$ and J was chosen maximally. As such, the “near” term may be rewritten as

$$\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} \left\| \Big(\sum_{\substack{s\in\mathbb{T}\\ |I_s|>2|J|}} \varepsilon_s \langle f, \psi_{s,f}\rangle \chi_{E\cap N^{-1}(\omega_s^+)} \psi_{s,f} \Big) \chi_{\tilde{J}} \right\|_{B^1}.$$

Define $G(J) := \tilde{J} \cap \bigcup_{\substack{s\in\mathbb{T}\\ |I_s|>2|J|}} (E\cap N^{-1}(\omega_s^+))$. To allow the desired bound for the “near” term to be proved, the following density estimate for $G(J)$ will be established:

$$\mu(G(J)) \lesssim \mathscr{M}(E;\mathbb{T})\mu(E)|J|.$$

To prove this, define J' to be the unique $\frac{1}{\alpha}$-dyadic parent of J and observe that by the maximal choice of J, there exists $s' \in \mathbb{T}$ such that $I_{s'} \subseteq 3 \star J'$. As it is a $\frac{1}{\alpha}$-dyadic interval, $I_{s'}$ is either exactly two thirds of $3 \star J'$ or it is contained in either the left or right third of $3 \star J'$. In the first case, define $s'' := s'$. In the second case, let $s'' \in \mathbb{D}_{0,\alpha}$ be chosen such that $|I_{s''}| = |J'|$ and $s' < s'' < \text{top}(\mathbb{T})$. Note that this choice of s'' is not necessarily in $\mathbb{T}$.

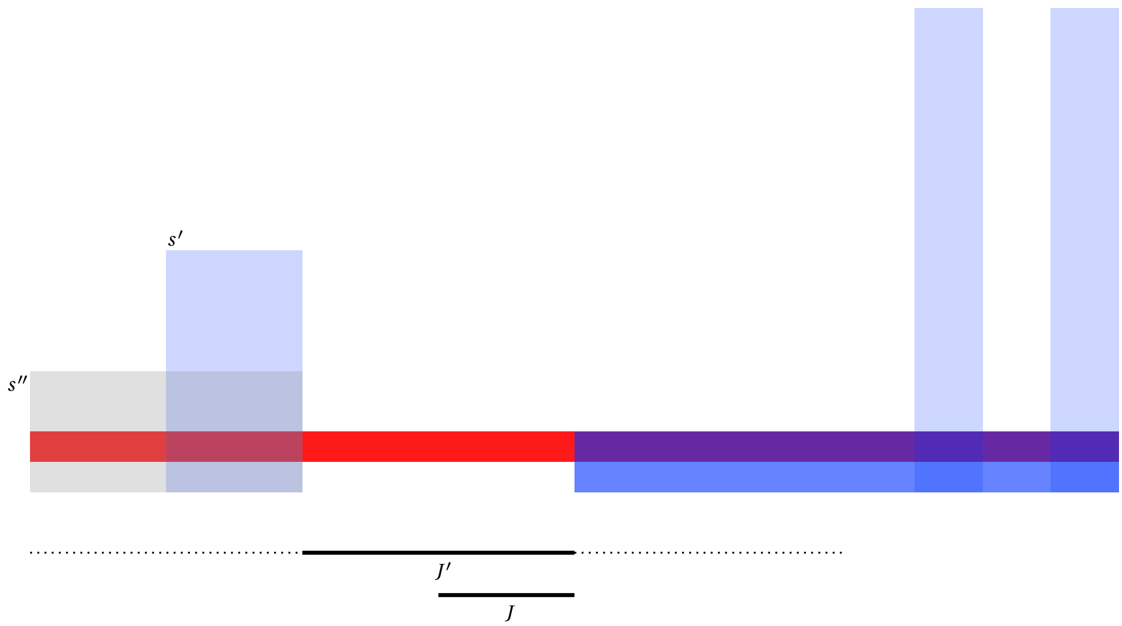

Figure 4.1 – An example of the process of selecting s' and s'': The fixed interval J and its dyadic parent J' are indicated as black lines. The dotted lines extending J' indicate $3 \star J'$. The tree $\mathbb{T}$ consists of the blue tiles together with the red tile as its top, although the faded blue tiles do not satisfy $|I_s| > 2|J|$ and hence do not contribute to the part of the "near" term under consideration. The grey selected tile, s'', is not a part of $\mathbb{T}$.

In both cases, it is claimed that for any tiles $s \in \mathbb{T}$ "near" J (that is $|I_s| > 2|J|$), it follows that $\omega_s \subseteq \omega_{s''}$. Indeed, to prove this, consider the alternatives:

- Suppose $\omega_{s''} \cap \omega_s = \emptyset$. Then as $s'' < \text{top}(\mathbb{T})$, $\omega_s \cap \omega_{\text{top}(\mathbb{T})} = \emptyset$, so $s \notin \mathbb{T}$, which is a contradiction.

- Suppose $\omega_{s''} \subsetneq \omega_s$. Then $|I_s| < |I_{s''}| \leqslant 4|J|$, which is also a contradiction, as it implies that $|I_s| \leqslant 2|J|$.

It follows that a tile $s'' \in \mathbb{D}_{0,\alpha}$ has been found that has a frequency projection covering all the frequency projections of tiles from the definition of $G(J)$ in the sense that $G(J) \subseteq \widetilde{J} \cap E \cap N^{-1}(\omega_{s''})$.

Using the definition of mass and making some straightforward estimates, it can be seen that

$$\begin{aligned}
& \mathscr{M}(E;\mathbb{T})\mu(E)|J| \\
\gtrsim\ & \mathscr{M}(E;\{s'\})\mu(E)|I_{s''}| \\
\geqslant\ & \lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap E\cap N^{-1}(\omega_{s''})}\sum_{m\in\mathbb{Z}}\frac{1}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_{s''})|}{|I_{s''}|}\right)^{10}}\,dx \\
\geqslant\ & \lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap\widetilde{J}\cap E\cap N^{-1}(\omega_{s''})}\sum_{m\in\mathbb{Z}}\frac{1}{\left(1+\frac{|x+\frac{m}{\alpha}-c(I_{s''})|}{|J|}\right)^{10}}\,dx \\
\geqslant\ & \lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap\widetilde{J}\cap E\cap N^{-1}(\omega_{s''})}\sup_{m\in\mathbb{Z}}\left(1+\frac{|x+\frac{m}{\alpha}-c(I_{s''})|}{|J|}\right)^{-10}\,dx.
\end{aligned}$$

For $x \in J$, given that $I_{s'} \subseteq 3\star J'$, $|x-c(I_{s'})| \leqslant 4|J|$. Additionally, $s' < s''$ and $|I_{s''}| \leqslant 4|J|$, so it follows that $|x-c(I_{s''})| \leqslant 8|J|$. Clearly, for any $x \in \widetilde{J}$, there exists $m \in \mathbb{Z}$ such that $x+\frac{m}{\alpha} \in J$, so

$$\lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap\widetilde{J}\cap E\cap N^{-1}(\omega_{s''})}\sup_{m\in\mathbb{Z}}\left(1+\frac{|x+\frac{m}{\alpha}-c(I_{s''})|}{|J|}\right)^{-10}\,dx \gtrsim \mu(\widetilde{J}\cap E\cap N^{-1}(\omega_{s''})).$$

Given that it was established that $G(J) \subseteq \widetilde{J}\cap E\cap N^{-1}(\omega_{s''})$, the desired estimate, $\mu(G(J)) \lesssim \mathscr{M}(E;\mathbb{T})\mu(E)|J|$ holds.

With this size estimate established, splitting the "near" term further, it can be seen that

$$\begin{aligned}
& \sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}\left\|\Big(\sum_{\substack{s\in\mathbb{T}\\ |I_s|>2|J|}}\varepsilon_s\langle f,\psi_{s,f}\rangle\chi_{E\cap N^{-1}(\omega_s^+)}\psi_{s,f}\Big)\chi_{\widetilde{J}}\right\|_{B^1} \\
\leqslant\ & \sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}\Bigg(\left\|\Big(\sum_{\substack{s\in\mathbb{T}-\\ |I_s|>2|J|}}\varepsilon_s\langle f,\psi_{s,f}\rangle\chi_{E\cap N^{-1}(\omega_s^+)}\psi_{s,f}\Big)\chi_{\widetilde{J}}\right\|_{B^1} \\
& +\left\|\Big(\sum_{\substack{s\in\mathbb{T}+\\ |I_s|>2|J|}}\varepsilon_s\langle f,\psi_{s,f}\rangle\chi_{E\cap N^{-1}(\omega_s^+)}\psi_{s,f}\Big)\chi_{\widetilde{J}}\right\|_{B^1}\Bigg),
\end{aligned}$$

where $\mathbb{T} = \mathbb{T}- \cup \mathbb{T}+$ and $\mathbb{T}-$ is a –tree and $\mathbb{T}+$ is a +tree.

These two terms will be considered separately. Looking at the –tree term first, observe that

$$\begin{aligned}
&\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} \Big\|\Big(\sum_{\substack{s\in\mathbb{T}-\\ |I_s|>2|J|}} \varepsilon_s\langle f,\psi_{s,f}\rangle \chi_{E\cap N^{-1}(\omega_s^+)}\psi_{s,f}\Big)\chi_{\tilde{J}}\Big\|_{B^1}\\
= &\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} \lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap\tilde{J}}\Big|\sum_{\substack{s\in\mathbb{T}-\\ |I_s|>2|J|}} \varepsilon_s\langle f,\psi_{s,f}\rangle \chi_{E\cap N^{-1}(\omega_s^+)}(x)\psi_{s,f}(x)\Big|\,dx\\
\leqslant &\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} \mu(G(J))\Big\|\sum_{\substack{s\in\mathbb{T}-\\ |I_s|>2|J|}} \varepsilon_s\langle f,\psi_{s,f}\rangle \chi_{N^{-1}(\omega_s^+)}\psi_{s,f}\Big\|_{L^\infty(\tilde{J})}.
\end{aligned}$$

By the –tree structure, for s and $s' \in \mathbb{T}-$ at different scales, $\omega_s^+ \cap \omega_{s'}^+ = \emptyset$, so a certain disjointness is enforced in the sum in s. Using the estimate on $\mu(G(J))$ and observing, as previously, that as the singleton $\{s\}$ is a +tree, the estimate $|\langle f,\psi_{s,f}\rangle| \leqslant |I_s|^{\frac{1}{2}}\|f\|_{B^2}\mathscr{E}(f;\mathbb{T})$ holds, it can thus be seen that the above expression is bounded by

$$\begin{aligned}
&\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} \mu(G(J)) \sup_{k>\log_2 2\alpha|J|}\Big\|\sum_{\substack{s\in\mathbb{T}-\\ |I_s|=\frac{2^k}{\alpha}}} |\langle f,\psi_{s,f}\rangle\psi_{s,f}|\Big\|_{L^\infty(\tilde{J})}\\
\leqslant\ &\mathscr{M}(E;\mathbb{T})\mu(E)\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} |J| \sup_{k>\log_2 2\alpha|J|}\frac{2^{\frac{k}{2}}}{\sqrt{\alpha}}\|f\|_{B^2}\mathscr{E}(f;\mathbb{T})\Big\|\sum_{\substack{s\in\mathbb{T}-\\ |I_s|=\frac{2^k}{\alpha}}} |\psi_{s,f}|\Big\|_{L^\infty(\tilde{J})}\\
\lesssim\ &\mathscr{E}(f;\mathbb{T})\mathscr{M}(E;\mathbb{T})\|f\|_{B^2}\mu(E)\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} |J| \sup_{k>\log_2 2\alpha|J|}\sup_{x\in\tilde{J}}\frac{1}{\sqrt{\alpha}}\sum_{\substack{s\in\mathbb{T}-\\ |I_s|=\frac{2^k}{\alpha}}}\sum_{m\in\mathbb{Z}}\frac{1}{(1+\frac{|x+\frac{m}{\alpha}-c(I_s)|}{|I_s|})^{10}}
\end{aligned}$$

by the estimate on $|\psi_{s,f}|$ from Proposition 2.3.3.

Now, in the sum in s, the I_s are disjoint and contained in $[0,\frac{1}{\alpha})$, so

$$\sum_{\substack{s\in\mathbb{T}-\\ |I_s|=\frac{2^k}{\alpha}}}\sum_{m\in\mathbb{Z}}\frac{1}{(1+\frac{|x+\frac{m}{\alpha}-c(I_s)|}{|I_s|})^{10}} \lesssim \sum_{n\in\mathbb{Z}}\frac{1}{(1+n)^{10}}.$$

It hence follows that the –tree term is bounded by a constant multiple of

$$\frac{1}{\sqrt{\alpha}}\mathscr{E}(f;\mathbb{T})\mathscr{M}(E;\mathbb{T})\|f\|_{B^2}\mu(E)\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}|J| \lesssim \frac{1}{\sqrt{\alpha}}|I_{\mathrm{top}(\mathbb{T})}|\mathscr{E}(f;\mathbb{T})\mathscr{M}(E;\mathbb{T})\|f\|_{B^2}\mu(E).$$

This gives the desired estimate on the –tree term. It remains to consider the +tree term:

$$\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}\left\|\Bigg(\sum_{\substack{s\in\mathbb{T}+\\ |I_s|>2|J|}}\varepsilon_s\langle f,\psi_{s,f}\rangle\chi_{E\cap N^{-1}(\omega_s^+)}\psi_{s,f}\Bigg)\chi_{\widetilde{J}}\right\|_{B^1}.$$

Observe that in a +tree, all the upper frequency projections are nested. For each x, this fact allows the tiles s for which $N(x)$ is in ω_s^+ to be more easily identified. Indeed, for each fixed $J\in\mathscr{J}$ with $J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}$, choose $u_x, v_x\in\mathbb{T}+\cap\{s\in\mathbb{D}_{0,\alpha}:|I_s|>2|J|\}$ for each $x\in\widetilde{J}$ such that for any $s\in\mathbb{T}+$ with $|I_s|>2|J|$,

$$N(x)\in\omega_s^+ \Leftrightarrow \omega_{v_x}\subseteq\omega_s\subseteq\omega_{u_x} \Leftrightarrow |\omega_{v_x}|\leqslant|\omega_s|\leqslant|\omega_{u_x}|.$$

Let $\gamma\in\mathscr{S}(\mathbb{R})$ be such that $\widehat{\gamma}(\xi)=0$ for $|\xi|>\frac{101}{100}$ and $\widehat{\gamma}(\xi)=1$ for $|\xi|\leqslant 1$. For $x,\lambda\in\mathbb{R}$, define function $\theta\in\mathscr{S}(\mathbb{R})$ to be such that

$$\widehat{\theta_x}(\lambda)=\widehat{\gamma}\left(\frac{\lambda-c(\omega_{u_x})}{|\omega_{u_x}|}\right)-\widehat{\gamma}\left(\frac{\lambda-c(\omega_{v_x}^+)}{|\omega_{v_x}^+|}\right).$$

It follows from this definition that if s is a tile in $\mathbb{T}+$ with $|I_s|>2|J|$ such that $|\omega_{v_x}|\leqslant|\omega_s|\leqslant|\omega_{u_x}|$ and $\lambda\in\mathbb{R}$ is such that $\widehat{\psi_{s,f}}(\lambda)\neq 0$ then $\widehat{\theta_x}(\lambda)=1$, whilst if s is any other tile in $\mathbb{T}+$ and $\lambda\in\mathbb{R}$ is such that $\widehat{\psi_{s,f}}(\lambda)\neq 0$, then $\widehat{\theta_x}(\lambda)=0$.

It follows, using Theorem 1.3.8, that the integrand from the +tree term can be written in terms of an operator defined by convolution with a Schwartz function. This in turn allows it to be bounded by a maximal function of Hardy–Littlewood type:

$$\begin{aligned}
&\Big|\sum_{\substack{s\in\mathbb{T}+\\|I_s|>2|J|}}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\chi_{E\cap N^{-1}(\omega_s^+)}(x)\psi_{s,f}(x)\Big|\\
=\;&\Big|\Big(\big(\sum_{s\in\mathbb{T}+}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}\big)*\theta_x\Big)(x)\chi_E(x)\Big|\\
\leqslant\;&2\chi_E(x)\sup_{\delta>|\omega_{u_x}|^{-1}}\int_{\mathbb{R}}\Big|\sum_{s\in\mathbb{T}+}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}(y)\Big|\frac{1}{\delta}\Big|\gamma\Big(\frac{x-y}{\delta}\Big)\Big|\,dy\\
\lesssim\;&\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbb{T}+}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}(y)\Big|\,dy.
\end{aligned}$$

By these deductions, the +tree term may be bounded as follows:

$$\begin{aligned}
&\sum_{\substack{J\in\mathscr{J}\\J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}\Big\|\Big(\sum_{\substack{s\in\mathbb{T}+\\|I_s|>2|J|}}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\chi_{E\cap N^{-1}(\omega_s^+)}\psi_{s,f}\Big)\chi_{\widetilde{J}}\Big\|_{B^1}\\
=\;&\sum_{\substack{J\in\mathscr{J}\\J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}\lim_{T\to\infty}\frac{1}{2T}\int_{[-T,T]\cap\widetilde{J}}\Big|\sum_{\substack{s\in\mathbb{T}+\\|I_s|>2|J|}}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\chi_{E\cap N^{-1}(\omega_s^+)}(x)\psi_{s,f}(x)\Big|\,dx\\
\lesssim\;&\sum_{\substack{J\in\mathscr{J}\\J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}\mu(G(J))\sup_{x\in\widetilde{J}}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbb{T}+}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}(y)\Big|\,dy\\
\lesssim\;&\mathscr{M}(E;\mathbb{T})\mu(E)\sum_{\substack{J\in\mathscr{J}\\J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}}|J|\sup_{x\in\widetilde{J}}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbb{T}+}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}(y)\Big|\,dy.
\end{aligned}$$

By the recentring properties of the Hardy–Littlewood maximal operator,

$$\begin{aligned}
&\sup_{x\in\widetilde{J}}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\Big|\sum_{s\in\mathbb{T}+}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}(y)\Big|\,dy\\
\lesssim\;&\inf_{x\in J}\sup_{k\in\mathbb{Z}}\sup_{\delta>4|J|}\frac{1}{2\delta}\int_{x+\frac{k}{\alpha}-\delta}^{x+\frac{k}{\alpha}+\delta}\Big|\sum_{s\in\mathbb{T}+}\varepsilon_s\langle\!\langle f,\psi_{s,f}\rangle\!\rangle\psi_{s,f}(y)\Big|\,dy.
\end{aligned}$$

Replacing the infimum above by an integral over J and then observing that this integral may be rewritten as an averaged integral over $\widetilde{J}$ by periodicity of the integrand, the +tree term is seen to be bounded by a constant multiple of

$$\mathscr{M}(E;\mathbb{T})\mu(E) \sum_{\substack{J\in\mathscr{J} \\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} \frac{1}{\alpha}\Big\|\widetilde{M}\Big(\sum_{s\in\mathbb{T}+} \varepsilon_s \langle f,\psi_{s,f}\rangle \psi_{s,f}\Big)\chi_{\tilde{J}}\Big\|_{B^1}$$

where $\widetilde{M}$ is a variant of the standard Hardy–Littlewood maximal operator, defined, for a suitable function g, as

$$\widetilde{M}g(x) \coloneqq \sup_{k\in\mathbb{Z}}\sup_{\delta>0} \frac{1}{2\delta}\int_{x+\frac{k}{\alpha_g}-\delta}^{x+\frac{k}{\alpha_g}+\delta} |g(y)|\,dy.^*$$

By Hölder's inequality, the +tree term is seen to be further bounded by a constant multiple of

$$\mathscr{M}(E;\mathbb{T})\mu(E)\frac{1}{\alpha}(\alpha|I_{\mathrm{top}(\mathbb{T})}|)^{\frac{1}{2}}\Big\|\widetilde{M}\Big(\sum_{s\in\mathbb{T}+} \varepsilon_s \langle f,\psi_{s,f}\rangle \psi_{s,f}\Big)\Big\|_{B^2}.$$

Additionally, recalling the definition of $\psi_{s,f}$ and using that $\widetilde{M}$ is a sub-linear operator, it can be seen that

$$\begin{aligned}
&\Big\|\widetilde{M}\Big(\sum_{s\in\mathbb{T}+} \varepsilon_s \langle f,\psi_{s,f}\rangle \psi_{s,f}\Big)\Big\|_{B^2} \\
= \;& \Big\|\widetilde{M}\Big(\sum_{s\in\mathbb{T}+} \varepsilon_s \langle f,\psi_{s,f}\rangle \frac{\sqrt{\alpha}}{|\Lambda_f|}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}}\sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n+\alpha m)e^{2\pi i(\lambda_n+\alpha m)\cdot}\Big)\Big\|_{B^2} \\
\leqslant \;& \frac{1}{|\Lambda_f|}\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}}\Big\|\widetilde{M}\Big(\sum_{s\in\mathbb{T}+} \varepsilon_s \langle f,\psi_{s,f}\rangle \sqrt{\alpha}\sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n+\alpha m)e^{2\pi i(\lambda_n+\alpha m)\cdot}\Big)\Big\|_{B^2}.
\end{aligned}$$

Since it is now operating on a function of period $\frac{1}{\alpha}$, the operator $\widetilde{M}$ may be replaced with the regular Hardy–Littlewood maximal operator, M, so using the boundedness of M as an operator on $L^2([0,\frac{1}{\alpha}))$, the above quantity is seen to be bounded by a constant multiple of

$$\max_{\lambda_n\in\Lambda_f}\Big\|\sum_{s\in\mathbb{T}+} \varepsilon_s \langle f,\psi_{s,f}\rangle \sqrt{\alpha}\sum_{m\in\mathbb{Z}} \widehat{\phi_s}(\lambda_n+\alpha m)e^{2\pi i(\lambda_n+\alpha m)\cdot}\Big\|_{B^2}.$$

*This operator will be discussed further in the appendix to this part.

Defining $g_\lambda(x) \coloneqq e^{2\pi i\lambda x} + e^{2\pi i(\lambda+\alpha)x}$, for example, this quantity may be rewritten as

$$\max_{\lambda_n\in\Lambda_f}\Big\| \sum_{s\in\mathbb{T}+} \varepsilon_s \langle\!\langle f,\psi_{s,f}\rangle\!\rangle \psi_{s,g_{\lambda_n}}\Big\|_{B^2}$$

so the +tree term is seen to be bounded by a constant multiple of

$$\frac{1}{\sqrt{\alpha}}\mathscr{M}(E;\mathbb{T})\mu(E)|I_{\mathrm{top}(\mathbb{T})}|^{\frac{1}{2}} \max_{\lambda_n\in\Lambda_f}\Big\| \sum_{s\in\mathbb{T}+} \varepsilon_s \langle\!\langle f,\psi_{s,f}\rangle\!\rangle \psi_{s,g_{\lambda_n}}\Big\|_{B^2}.$$

Following the same steps as in the proof of the energy lemma, it can be seen that for any $\lambda_n \in \Lambda_f$,

$$\begin{aligned}
&\Big\| \sum_{s\in\mathbb{T}+} \varepsilon_s \langle\!\langle f,\psi_{s,f}\rangle\!\rangle \psi_{s,g_{\lambda_n}}\Big\|_{B^2}^2 \\
\leqslant\ & \sum_{\substack{s,s'\in\mathbb{T}+\\ \omega_s=\omega_{s'}}} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle \langle\!\langle \psi_{s,g_{\lambda_n}},\psi_{s',g_{\lambda_n}}\rangle\!\rangle \langle\!\langle f,\psi_{s',f}\rangle\!\rangle| \\
\leqslant\ & \sum_{s\in\mathbb{T}+} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|^2 \sum_{\substack{s'\in\mathbb{T}+\\ \omega_s=\omega_{s'}}} |\langle\!\langle \psi_{s,g_{\lambda_n}},\psi_{s',g_{\lambda_n}}\rangle\!\rangle| \\
\lesssim\ & \sum_{s\in\mathbb{T}+} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|^2 \sum_{\substack{s'\in\mathbb{T}+\\ \omega_s=\omega_{s'}}} \left(\frac{|I_s|}{|I_{s'}|}\right)^{\frac{1}{2}} \sum_{l\in\mathbb{Z}} \int_{I_{s'}+\frac{l}{\alpha}} \frac{|I_s|^{-1}\,dx}{(1+\frac{|x-c(I_s)|}{|I_s|})^{10}} \\
\lesssim\ & \sum_{s\in\mathbb{T}+} |\langle\!\langle f,\psi_{s,f}\rangle\!\rangle|^2 \\
\leqslant\ & |I_{\mathrm{top}(\mathbb{T})}| \mathscr{E}(f;\mathbb{T})^2 \|f\|_{B^2}^2.
\end{aligned}$$

Overall, it may thus be concluded that

$$\sum_{\substack{J\in\mathscr{J}\\ J\subseteq 3\star I_{\mathrm{top}(\mathbb{T})}}} \Big\|\Big(\sum_{\substack{s\in\mathbb{T}+\\ |I_s|>2|J|}} \varepsilon_s \langle\!\langle f,\psi_{s,f}\rangle\!\rangle \chi_{E\cap N^{-1}(\omega_s^+)}\psi_{s,f}\Big)\chi_{\tilde{J}}\Big\|_{B^1} \lesssim \frac{1}{\sqrt{\alpha}} |I_{\mathrm{top}(\mathbb{T})}| \mathscr{E}(f;\mathbb{T}) \mathscr{M}(E;\mathbb{T}) \|f\|_{B^2}\mu(E),$$

which completes the proof of the tree lemma. $\square$

Some Technical Details

On page 112, in establishing the inequality

$$\sum_{J \in \mathscr{J}} \sum_{\substack{s \in \mathbb{T} \\ |I_s| \leqslant 2|J|}} |I_s| \sup_{x \in \widetilde{J}} \left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^{-10} \lesssim |I_{\mathrm{top}(\mathbb{T})}|,$$

it was assumed that the supremum over $\widetilde{J}$ could be replaced with a supremum over J. Whilst this is essentially true, there are circumstances under which the supremum can be attained for $x \in J \pm \frac{1}{\alpha}$. This introduces some small technical issues, the resolution of which is provided now.

Firstly, assume that $|J| \neq \frac{1}{4\alpha}$ for all $J \in \mathscr{J}$. The case where there are $J \in \mathscr{J}$ with $|J| = \frac{1}{4\alpha}$ will be dealt with separately.

Proceeding in a similar manner to before,

$$\begin{aligned}
&\sum_{J \in \mathscr{J}} \sum_{\substack{s \in \mathbb{T} \\ |I_s| \leqslant 2|J|}} |I_s| \sup_{x \in \widetilde{J}} \left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^{-10} \\
= &\sum_{J \in \mathscr{J}} \sum_{k=-\infty}^{\log_2 2\alpha|J|} \frac{2^k}{\alpha} \sum_{\substack{s \in \mathbb{T} \\ |I_s| = \frac{2^k}{\alpha}}} \sup_{x \in \widetilde{J}} \left(1 + \frac{|x - c(I_s)|}{|I_s|}\right)^{-10} \\
\leqslant &\sum_{J \in \mathscr{J}} \sum_{k=-\infty}^{\log_2 2\alpha|J|} \frac{2^k}{\alpha} \sup_{x \in \widetilde{J}} \left(1 + \frac{\mathrm{dist}(x, I_{\mathrm{top}(\mathbb{T})})}{|I_{\mathrm{top}(\mathbb{T})}|}\right)^{-5} \sum_{\substack{s \in \mathbb{T} \\ |I_s| = \frac{2^k}{\alpha}}} \left(\left(1 + \frac{\mathrm{dist}(J, I_s)}{\alpha^{-1} 2^k}\right)^{-5} \right. \\
&\quad \left. + \left(1 + \frac{\mathrm{dist}(J + \frac{1}{\alpha}, I_s)}{\alpha^{-1} 2^k}\right)^{-5} + \left(1 + \frac{\mathrm{dist}(J - \frac{1}{\alpha}, I_s)}{\alpha^{-1} 2^k}\right)^{-5} \right).
\end{aligned}$$

It is still the case that

$$\sum_{\substack{s \in \mathbb{T} \\ |I_s| = \frac{2^k}{\alpha}}} \left(\left(1 + \frac{\mathrm{dist}(J, I_s)}{\alpha^{-1} 2^k}\right)^{-5} + \left(1 + \frac{\mathrm{dist}(J + \frac{1}{\alpha}, I_s)}{\alpha^{-1} 2^k}\right)^{-5} + \left(1 + \frac{\mathrm{dist}(J - \frac{1}{\alpha}, I_s)}{\alpha^{-1} 2^k}\right)^{-5} \right) \lesssim 1$$

so it remains to show that

$$\sum_{J\in\mathscr{J}}\sum_{k=-\infty}^{\log_2 2\alpha|J|}\frac{2^k}{\alpha}\sup_{x\in\widetilde{J}}\left(1+\frac{\operatorname{dist}(x,I_{\mathrm{top}(\mathbb{T})})}{|I_{\mathrm{top}(\mathbb{T})}|}\right)^{-5}\lesssim|I_{\mathrm{top}(\mathbb{T})}|.$$

If the supremum here can indeed be replaced by a supremum over J, then working as before, this result will follow. As such, it suffices to show that for any $J\in\mathscr{J}$ with $|J|\neq\frac{1}{4\alpha}$, it is necessarily the case that

$$\operatorname{dist}(J\pm\tfrac{1}{\alpha},I_{\mathrm{top}(\mathbb{T})})\geqslant\operatorname{dist}(J,I_{\mathrm{top}(\mathbb{T})}).$$

To prove this claim, first note that it cannot be the case that $|J|=\frac{1}{\alpha}$ or $|J|=\frac{1}{2\alpha}$ as this would necessitate that $\mathbb{T}$ be empty, owing to the way that the intervals in $\mathscr{J}$ were selected. It can thus be assumed that $|J|<\frac{1}{4\alpha}$. Also, if J and $I_{\mathrm{top}(\mathbb{T})}$ overlap, then $\operatorname{dist}(J,I_{\mathrm{top}(\mathbb{T})})=0$ and the claim follows, so it will suffice to consider when they are disjoint.

Divide the time window, $[0,\frac{1}{\alpha})$, into two halves, each of which is formed of an innermost and outermost quarter. If J and $I_{\mathrm{top}(\mathbb{T})}$ are both contained in a single half, the claim follows. If they are in separate halves, by the maximal choice of J as well as its $\frac{1}{\alpha}$-dyadic size, J must be contained in the inner quarter of its half and $I_{\mathrm{top}(\mathbb{T})}$ must either be the whole of the half it is contained in or be contained in the inner quarter of its half. In either case, the claim follows.

For J with $|J|=\frac{1}{4\alpha}$, this inequality can fail*, so this situation needs to be dealt with separately. Since there are at most four $J\in\mathscr{J}$ satisfying $|J|=\frac{1}{4\alpha}$, it suffices to show that for any such J,

$$\sum_{\substack{s\in\mathbb{T}\\|I_s|\leqslant 2|J|}}|I_s|\sup_{x\in\widetilde{J}}\left(1+\frac{|x-c(I_s)|}{|I_s|}\right)^{-10}\lesssim|I_{\mathrm{top}(\mathbb{T})}|$$

*For example, it is possible to let $J=[0,\frac{1}{4\alpha})$ and have $I_{\mathrm{top}(\mathbb{T})}$ contained in $[\frac{3}{4\alpha},\frac{1}{\alpha})$.

and any remaining intervals in $\mathscr{J}$ can be dealt with as previously. However, noting that the left hand side of the above is bounded by

$$\sum_{k=-\infty}^{\log_2 \alpha |I_{\mathrm{top}(\mathbb{T})}|} \frac{2^k}{\alpha} \sum_{\substack{s\in\mathbb{T} \\ |I_s|=\frac{2^k}{\alpha}}} \left(\left(1+\frac{\mathrm{dist}(J,I_s)}{\alpha^{-1}2^k}\right)^{-10} + \left(1+\frac{\mathrm{dist}(J+\frac{1}{\alpha},I_s)}{\alpha^{-1}2^k}\right)^{-10} + \left(1+\frac{\mathrm{dist}(J-\frac{1}{\alpha},I_s)}{\alpha^{-1}2^k}\right)^{-10} \right),$$

the result follows as before.

CONCLUDING REMARKS FOR PART I

Completion of the Proposed Proof

The work presented in this part of this thesis is a record of partial progress towards a proof of Conjecture 3.5.1 (or some closely analogous conjecture), leading to the establishment of time-frequency techniques adequate to prove the almost periodic form of Carleson's Theorem (stated as Theorem 2.1.1) directly. The established main estimate in Theorem 3.5.2 falls short of Conjecture 3.5.1 with respect to the power of $\frac{1}{|\Lambda_f|}$ from the estimate therein, which is not yet sufficiently large to counterbalance the implicit factor of $\frac{1}{|\Lambda_f|^2}$ from the definition of the model operator. In spite of the fact that Conjecture 3.5.1 remains open, it has been shown that a reasonable analogue of the scheme of Lacey and Thiele can be constructed in the almost periodic setting and also that many aspects of the analysis of Lacey and Thiele can be made to work for this analogue. Indeed, a model operator has been proposed which, after an averaging process similar to that of Lacey and Thiele, has been shown to be equivalent to the almost periodic Fourier summation operator in a natural way, in that the averaged operator, Π_ξ, satisfies the equivalence

$$\frac{1}{|\Lambda_f|^2}\mathscr{C} f \approx \sup_{\xi\in\Xi_f} |\Pi_\xi f|$$

for $f \in \mathscr{P}$, where

$$\Xi_f := \{\xi \in \mathbb{R} : \xi - \lambda_n > \tfrac{2\alpha_f}{5}\ \forall \lambda_n \in \sigma(f) \cap (-\infty, \xi]\}.$$

At the very least, this equivalence can be used to conclude that all known bounds for the almost periodic Carleson operator (which can be deduced from the bounds on the Carleson operator on $L^p(\mathbb{R})$ functions using the transference results of Section 1.3.3) must also hold for the operator $|\Lambda_f|^2 \sup_{\xi\in\Xi_f} |\Pi_\xi \cdot|$.

In addition, the fact that the proposed model operator is susceptible to combinatorial time-frequency analysis similar to that used by Lacey and Thiele to bound their model operator can be seen from the fact that natural analogues of mass and energy in this context have been formulated and shown to satisfy lemmata analogous to those of Lacey and Thiele as well as relate to the model operator in an appropriate way.

The difficulty in establishing the higher power of $\frac{1}{|\Lambda_f|}$ in the main estimate required to conclude Conjecture 3.5.1 is that powers of $\frac{1}{|\Lambda_f|}$ are lost when the estimates of Proposition 2.3.3 are used (in the proof of these estimates, this occurs when sums over Λ_f are bounded by the cardinality of Λ_f multiplied by the largest element from the sum). Improvements of the estimates of Proposition 2.3.3 as stated seem unlikely. Indeed, it seems reasonable to expect $|\psi_{s,f}|$ to be potentially badly behaved in at least small regions, due to the way that linearly independent oscillations in its definition can interfere constructively. As such, uniform pointwise bounds in this setting are unlikely to be "good". Further, by applying Parseval's identity*, it also seems reasonable to expect the expression $|\Lambda_f|^2\langle\psi_{s,f},\psi_{s',f}\rangle$ to grow in a roughly linear manner as $|\Lambda_f|$ is increased. This fact is made particularly clear by considering that, as was shown in the proof of Proposition 2.3.3,

$$|\langle\psi_{s,f},\psi_{s',f}\rangle| = \frac{1}{|\Lambda_f|^2}\Big|\sum_{\substack{n\in\mathbb{Z}\\ \lambda_n\in\Lambda_f}}\sum_{l\in\mathbb{Z}}\langle\phi_{I_s\times(\omega_s-\lambda_n)},\phi_{(I_{s'}-\frac{l}{\alpha_f})\times(\omega_{s'}-\lambda_n)}\rangle_{L^2(\mathbb{R})}\Big|,$$

and it can be seen that the terms inside the sum in n here do not depend on n, since for an arbitrary tile s, the function ϕ_s was defined as

$$\phi_s(x) := |I_s|^{-\frac{1}{2}}\phi\left(\frac{x-c(I_s)}{|I_s|}\right)e^{2\pi i c(\omega_s^-)x}.$$

*Here, "Parseval's identity" can be interpreted as meaning either the identity $\|f\|_{B^2} = \left(\sum_{n\in\mathbb{Z}}|\widehat{f}(\lambda_n)|^2\right)^{\frac{1}{2}}$ for $f\in B^2$, as stated in Theorem 1.3.7, or the more general identity, $\langle f,g\rangle = \sum_{n\in\mathbb{Z}}\widehat{f}(\lambda_n)\overline{\widehat{g}(\lambda_n)}$, which can easily be deduced for functions $f, g\in\mathscr{P}$ using orthonormality arguments, as used in various proofs in this part of this thesis.

In contrast with attempting to improve the estimates of Proposition 2.3.3, a more likely route to improving the power of $\frac{1}{|\Lambda_f|}$ in the main estimate seems to be increasing the sophistication of the parts of the proof of Theorem 3.5.2 that use Proposition 2.3.3, exploiting averaged properties of the localising function *in situ* rather than relying on pointwise bounds and inner product estimates. Specifically, the proof of the main estimate makes use of these estimates in the proofs of both the energy lemma and the tree lemma and it is here where it is proposed that an improvement should be sought. Regardless of how they are arrived at, it seems likely that the only route to establishing Conjecture 3.5.1 within the current scheme is by proving the following improved versions:

The Energy Conjecture *Let f be a trigonometric polynomial and let $\mathbb{P}_f$ be an arbitrary finite collection of tiles from $\mathbb{D}_{0,\alpha_f}$. Then $\mathbb{P}_f$ can be written as $\mathbb{P}_f^{\text{low}} \sqcup \mathbb{P}_f^{\text{high}}$ where*

$$\mathscr{E}(f;\mathbb{P}_f^{\text{low}}) \leqslant \frac{1}{2}\mathscr{E}(f;\mathbb{P}_f)$$

and $\mathbb{P}_f^{\text{high}}$ is a union of trees $\mathbb{T}_j$ such that

$$\sum_j |I_j| \lesssim \frac{1}{|\Lambda_f|^2}\mathscr{E}(f;\mathbb{P}_f)^{-2}.$$

The Tree Conjecture *Let f be a trigonometric polynomial, let $E \subseteq \mathbb{R}$ be a measurable set such that $\mu(E)$ exists and is non-zero and let $N : \mathbb{R} \to \mathbb{R}$ be an arbitrary measurable function. Then for any tree, $\mathbb{T} \subseteq \mathbb{D}_{0,\alpha_f}$,*

$$\sum_{s\in\mathbb{T}} |\langle\!\langle (\chi_{\omega_s^+}\circ N)\psi_{s,f},\chi_E\rangle\!\rangle\langle\!\langle f,\psi_{s,f}\rangle\!\rangle| \lesssim \frac{1}{\sqrt{\alpha_f}|\Lambda_f|}|I_{\text{top}(\mathbb{T})}|\mathscr{E}(f;\mathbb{T})\mathscr{M}(E;\mathbb{T})\|f\|_{B^2}\mu(E).$$

If both of these conjectures were shown to be true, Conjecture 3.5.1 would follow and the proof of Theorem 2.1.1 would be complete. Of course, it is also possible that these results cannot be proved as they are currently stated and that some alteration to the current scheme for time-frequency localisation and formation of the model operator is required. It would seem prudent to initially consider this possibility within the simpler scope of non-maximal operators, reconsidering Theorem 3.1.2 which established that for functions $f \in \mathscr{P}$, the model operator satisfies the bound

$$\|A_{\rho,\xi} f\|_{B^2} \lesssim \frac{1}{|\Lambda_f|^{\frac{3}{2}}} \|f\|_{B^2}$$

with constant uniform in $\rho \in \mathbb{R}$ and $\xi \in \mathbb{R}$. Given that the averaged model operator is essentially equivalent to the almost periodic Fourier summation operator multiplied by $\frac{1}{|\Lambda_f|^2}$, it seems reasonable to expect the improved bound

$$\|A_{\rho,\xi} f\|_{B^2} \lesssim \frac{1}{|\Lambda_f|^2} \|f\|_{B^2}.$$

The "loss" of $\frac{1}{|\Lambda_f|^{\frac{1}{2}}}$ here is again due to a use of Proposition 2.3.3.

An alternative desirable goal, which is potentially less demanding than improving the power of $\frac{1}{|\Lambda_f|}$ in Theorem 3.1.2, is attempting to establish the bound

$$\|\Pi_\xi f\|_{B^2} \lesssim \frac{1}{|\Lambda_f|^2} \|f\|_{B^2}$$

by using the bound from Theorem 3.1.2. Indeed, this bound is immediate via the established equivalence with almost periodic Fourier summation (for "most" choices of ξ), but if the proposed scheme for bounding the Carleson operator, which involves transitioning from the maximal averaged model operator to the maximal model operator, is viable, the same scheme should certainly be viable in the less complicated, "non-maximal" setting. With respect to this goal, it is noted that the averaging process, as detailed in Section 3.3, effects much cancellation on the model operator owing to orthogonality properties of almost periodic exponentials; that this is the case is seen most clearly from the calculations in Section 3.3.3. With careful thought, the choices of Fourier exponents contained within the localising function may be subject to some modification without affecting the averaged operator. This might, for example, lead to an alternative model operator, $\widetilde{A}_{\rho,\xi}$, which corresponds to the same averaged operator as the regular model operator, but satisfies

$$\|\widetilde{A}_{\rho,\xi} f\|_{B^2} \lesssim \frac{1}{|\Lambda_f|^{\frac{1}{2}}} \|A_{\rho,\xi} f\|_{B^2}$$

for each ρ, $\xi \in \mathbb{R}$ and $f \in \mathscr{P}$. Taking this idea further, it might be possible to carry out a similar argument where the averaged operator corresponding to the altered model operator is not identical to the original, but is sufficiently close as to allow the desired bound to be established in this way.

Possible Extensions and Further Work

Given a complete version of the proposed proof of Carleson's Theorem on B^2, there are a number of natural additional questions that present themselves. Perhaps most apparent is that the extension of Lacey and Thiele's methods to $L^p(\mathbb{R})$ for $p \in (1, \infty)$ by Grafakos, Tao and Terwilleger[68] suggests the natural problem of extending the present work to the B^p spaces for $p \in (1, \infty)$. That paper, together with its predecessor by Pramanik and Terwilleger[119], also considers higher-dimensional time-frequency techniques on $L^p(\mathbb{R}^n)$ and it is possible that a generalisation to higher dimensional techniques may be possible in the almost periodic setting also.

Additionally, as remarked in Section 2.1, at present, nothing has been said about what kind of convergence of almost periodic Fourier series might follow from weak B^2 boundedness of the Carleson operator. Given that functions differing on sets of infinite measure may be equivalent in B^2 norm (and that it is clear that equivalent functions have identical Fourier series), the question of pointwise convergence requires careful consideration in this setting. If it can be proved that Fourier series for functions in B^2 converge almost everywhere to *some* function, it may be that this function defines a canonical way for selecting a preferred representation of equivalent B^2 functions.

Adapting the standard L^p methods for showing almost everywhere convergence results from boundedness of maximal operators to the setting of Besicovitch spaces seems to be difficult. One possible route to showing pointwise convergence results for Fourier series for functions in B^2 from Carleson's theorem is to attempt to develop an analogue of the theorem of Katznelson mentioned in Section 1.1 which established that for $p \in (1, \infty)$, either the Fourier series of every $L^p(\mathbb{T})$ function converges almost everywhere or there exists a function in $L^p(\mathbb{T})$ whose Fourier series diverges everywhere. If a result of this type could be shown for B^p, it would significantly reduce the scope of the required convergence theorem; a direct analogue would require only that convergence of Fourier series could be shown at any single point to conclude a full almost everywhere convergence theorem.

With respect to pointwise convergence of almost periodic Fourier series, it is noted that the author's previous thesis[11] proved almost everywhere convergence of dyadic partial sums of Fourier series of almost periodic functions in the Stepanov spaces S^{2^k} $(k \in \mathbb{N})$ that satisfy the separation condition.* The proof given there proceeds by bounding the maximal operator†

$$\sup_{k \in \mathbb{N}} \Big| \sum_{|\lambda_n| \leqslant 2^k} \widehat{f}(\lambda_n) e^{2\pi i \lambda_n \cdot} \Big|$$

from S^{2^k} to S^{2^k}. This is done by bounding this maximal operator by a smoother maximal operator and a square function. Estimating each of these involves generalising boundedness of the Hardy–Littlewood maximal operator and the Hilbert transform to the Stepanov spaces, as well as establishing a result of Littlewood–Paley type in this setting. Pointwise almost everywhere convergence is deduced from boundedness of maximal operators without too much difficulty in this setting so it is natural to ask whether the proposed techniques for proving Carleson's Theorem in the Besicovitch spaces could be adapted to the Stepanov spaces. This question is particularly reasonable given that the Stepanov spaces are subspaces of the Besicovitch spaces. A resolution of this problem would allow a genuine full pointwise almost

*This work is also detailed in the author's paper, [10].

†To adhere to the conventions followed in the present thesis, the operator given here has been renormalised compared to the definition given in [11].

everywhere convergence result to be established for almost periodic Fourier series, although a direct adaptation of the techniques used for B^2 is complicated by the fact that the norm on the Stepanov space S^2 does not admit a natural inner product as the B^2 norm does.

The reverse question that presents itself in light of the work contained in this thesis is the extent to which the methods used to establish boundedness of the maximal dyadic Fourier summation operator in [11] can be generalised from the Stepanov spaces to the Besicovitch spaces. Whilst boundedness of the Hardy–Littlewood maximal operator appears to have been established on B^p for $p \in (1,\infty)$ in [69], and boundedness of the Hilbert transform on B^p for $p \in (1,\infty)$ can be deduced from the transference results of Section 1.3.3, the validity of a theory of Littlewood–Paley type would seem to be an open problem. Although boundedness of the Carleson operator on all B^p for $p \in (1,\infty)$ would include boundedness of the maximal dyadic Fourier summation operator as a special case, a proof of the form followed in [11] would almost certainly be significantly more concise. Additionally, results of Littlewood–Paley type, as well as a direct proof of boundedness of the Hilbert transform, would be of interest for their own sake and for the sake of further applications.

With respect to this latter question, it is noted that the techniques used to bound the Hilbert transform in [11] adapt without modification to the Besicovitch spaces, B^{2^k}, $k \in \mathbb{N}$.* Given that the B^p spaces are isometrically isomorphic to the spaces $L^p(\mathbb{R}_B, \mu_B)$ where $\mathbb{R}_B$ is the Bohr compactification of $\mathbb{R}$ and μ_B is its associated Haar measure, standard interpolation theory is valid in the Besicovitch spaces and so boundedness of the Hilbert transform can be deduced on B^p for all $p \in (1,\infty)$. However, no interpolation mechanism seems to be available in the Stepanov spaces and so boundedness of the Hilbert transform on S^p for general $p \in (1,\infty)$ remains open.† As was commented in the conclusion of [11], this is the only part of the proof of boundedness of the maximal dyadic Fourier summation operator that is only valid for S^{2^k}, $k \in \mathbb{N}$, rather than S^p, $p \in (1,\infty)$, and thus generalising the established bounds

*It should be clarified that these techniques are not time-frequency theoretic.

†As was mentioned in [11], a claimed proof of a stronger result than this did appear in [81]. However, the theorem claimed is provably false; the proposed proof contains an elementary error that does not seem to be circumventable, even to establish the weaker bound desired here.

on the Hilbert transform would also generalise the bounds on the maximal dyadic Fourier summation operator.

It is remarked that the case of B^2 for the problem of boundedness of the Hilbert transform on the Besicovitch spaces is particularly straightforward as it follows as an immediate consequence of Parseval's identity (this is actually more straightforward than in the Stepanov space setting since an exact Parseval-type identity does not exist for S^2). Nonetheless, given that the Fourier partial sum operators can be expressed as a combination of modulated Hilbert transforms and the identity operator, a closer study of the Hilbert transform on B^2, perhaps from a maximal perspective, could potentially offer some insight into time-frequency analysis of the Carleson operator on B^2; such insight may assist in further development of the work contained in this part of this thesis.

Appendix – A Maximal Operator of Hardy–Littlewood Type

The proof of the tree lemma considered very particular B^2 bounds for a generalisation of the Hardy–Littlewood maximal operator, defined as

$$\widetilde{M}f(x) \coloneqq \sup_{k\in\mathbb{Z}} \sup_{\delta>0} \frac{1}{2\delta} \int_{x+\frac{k}{\alpha_f}-\delta}^{x+\frac{k}{\alpha_f}+\delta} |f(y)|\, dy.$$

The purpose of this appendix is to show that this operator is not well-behaved on B^2 in general. Indeed, whilst it has been shown in [69] that the regular Hardy–Littlewood maximal operator is bounded on B^p for $p \in (1,\infty)$, the following also holds:

Theorem *Acting on $\mathscr{P}$, the operator $\widetilde{M}$ fails to be B^2 bounded.*

Proof For any $N \in \mathbb{N}\backslash\{1,2\}$, define $f_N(x) = \sum_{n=1}^{N} e^{2\pi^n ix}$. It is claimed that

$$\inf_{x\in\mathbb{R}} \sup_{k\in\mathbb{Z}} \left| f_N\left(x + \frac{k}{\alpha_{f_N}}\right)\right| \geqslant N-2$$

for any such N. Given this claim, note that

$$\|\widetilde{M}f_N\|_{B^2} \geqslant \inf_{x\in\mathbb{R}} \sup_{k\in\mathbb{Z}} \left| f_N\left(x + \frac{k}{\alpha_{f_N}}\right)\right| \geqslant N-2.$$

Observing that $\|f_N\|_{B^2} = \left(\sum_{n=1}^{N} 1^2\right)^{\frac{1}{2}} = \sqrt{N}$, by Parseval's Theorem, the result follows. It thus suffices to prove the claim. To do this, first consider that for each $k\in\mathbb{Z}$,

$$\begin{aligned}
&\left|\sum_{n=1}^{N} e^{2\pi^n i(x+\frac{k}{\alpha_{f_N}})}\right| \\
= \quad &\left|\sum_{n=1}^{N} e^{2\pi^n ix} e^{2\pi ik\frac{\pi^{n-1}}{\pi-1}}\right| \\
= \quad &\left| e^{2\pi ix} e^{2\pi ik\frac{1}{\pi-1}} + \sum_{n=2}^{N} e^{2\pi^n ix} e^{2\pi ik(\pi^{n-2}+\pi^{n-3}+\dots+\pi+1+\frac{1}{\pi-1})}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| e^{2\pi i k \frac{1}{\pi-1}} \left(e^{2\pi i x} + \sum_{n=2}^{N} e^{2\pi^n i x} e^{2\pi i k(\pi^{n-2}+\cdots+\pi+1)} \right) \right| \\
&= \left| e^{2\pi i x} + e^{2\pi^2 i x} + \sum_{n=3}^{N} e^{2\pi^n i x} e^{2\pi i k(\pi^{n-2}+\cdots+\pi)} \right| \\
&= \left| e^{2\pi i x} + e^{2\pi^2 i x} + \sum_{n=3}^{N} e^{2\pi^n i x} e^{2\pi^2 i k \frac{\pi^{n-2}-1}{\pi-1}} \right|.
\end{aligned}
$$

To motivate the proof, the base case of $N = 3$ will first be considered, that is to say that it will be shown that

$$\sup_{k\in\mathbb{Z}} |e^{2\pi i x} + e^{2\pi^2 i x} + e^{2\pi^3 i x + 2\pi^2 i k}| \geqslant 1$$

for all $x \in \mathbb{R}$.

It is claimed that $k \in \mathbb{Z}$ can be chosen so that $2\pi^3 x + 2\pi^2 k$ is arbitrarily close to $\arg(e^{2\pi i x} + e^{2\pi^2 i x})$ modulo 2π.* Indeed, this is equivalent to choosing k arbitrarily close to

$$\frac{\arg(e^{2\pi i x} + e^{2\pi^2 i x}) - 2\pi^3 x}{2\pi^2}$$

modulo $\frac{1}{\pi}$ and an expression of the form $k + \frac{l}{\pi}$ for $k, l \in \mathbb{Z}$ can approximate any real number arbitrarily well (see, for example, Dirichlet's Approximation Theorem on p. 143 of [5]), so the result follows for $N = 3$.

To establish the claim for general N, the following refinement of Dirichlet's Approximation Theorem will be used:

Kronecker's Theorem *Let $\alpha_1, \ldots, \alpha_n$ be arbitrary real numbers and let $\{\theta_1, \ldots, \theta_n, 1\} \subseteq \mathbb{R}$ be linearly independent over $\mathbb{Z}$. Then for any $\varepsilon > 0$, there exist $k, m_1, \ldots, m_n \in \mathbb{Z}$ such that*

$$|k + \frac{m_i}{\theta_i} - \alpha_i| < \varepsilon$$

for all $i \in [1, n] \cap \mathbb{N}$.

*The value assigned to $\arg(0)$ is unimportant here.

For a proof of this result, see [5, Ch. 7], for example.*

The desired inequality is that

$$\sup_{k\in\mathbb{Z}} |e^{2\pi ix} + e^{2\pi^2 ix} + e^{2\pi^3 ix + 2\pi^2 ik} + \ldots + e^{2\pi^N ix + 2\pi^2 ik \frac{\pi^{N-2}-1}{\pi-1}}| \geqslant N-2$$

for all $x \in \mathbb{R}$.

In the spirit of the $N = 3$ proof, this inequality will follow from showing that the expressions

$$\begin{cases} 2\pi^N x + 2\pi^2 \frac{\pi^{N-2}-1}{\pi-1} k \\ 2\pi^{N-1} x + 2\pi^2 \frac{\pi^{N-3}-1}{\pi-1} k \\ \vdots \\ 2\pi^4 x + 2\pi^2 \frac{\pi^2-1}{\pi-1} k \\ 2\pi^3 x + 2\pi^2 k \end{cases}$$

approximate $\arg(e^{2\pi ix} + e^{2\pi^2 ix})$ uniformly and arbitrarily well modulo 2π for suitable choices of k. However, this statement is equivalent to the conclusion of Kronecker's Theorem with

$$\begin{aligned} \alpha_1 &= \frac{\arg(e^{2\pi ix}+e^{2\pi^2 ix})-2\pi^3 x}{2\pi^2}, & \theta_1 &= \pi, \\ \alpha_2 &= \frac{(\arg(e^{2\pi ix}+e^{2\pi^2 ix})-2\pi^4 x)(\pi-1)}{2\pi^2(\pi^2-1)}, & \theta_2 &= \frac{\pi(\pi^2-1)}{\pi-1}, \\ &\vdots & &\vdots \\ \alpha_{N-2} &= \frac{(\arg(e^{2\pi ix}+e^{2\pi^2 ix})-2\pi^N x)(\pi-1)}{2\pi^2(\pi^{N-2}-1)}, & \theta_{N-2} &= \frac{\pi(\pi^{N-2}-1)}{\pi-1}. \end{aligned}$$

Observing that these θ_i do satisfy the hypotheses of Kronecker's Theorem, the result follows.

□

*The theorem as stated is a modified form of Theorem 7.10 from [5] (Second form of Kronecker's Theorem), stated on p. 154.

Table of Notation Used in Part I

The table below lists the non-standard notation that is used in the first part of this thesis with a brief explanation of each symbol and a reference to the page on which it is first defined. The reader is additionally referred to the remarks on notation at the beginning of the thesis for certain matters of notational disambiguation.

$\mathbb{D}$	The collection of all dyadic tiles in the time-frequency plane	p. 22
$\mathscr{P}$	The set of almost periodic trigonometric polynomials	p. 33
B^p	The Besicovitch almost periodic function spaces	p. 34
$\langle\!\langle \cdot, \cdot \rangle\!\rangle$	The B^2 inner product	p. 34
$\sigma(\cdot)$	The spectrum of an almost periodic function: $\sigma(f) \coloneqq \{\lambda \in \mathbb{R} : \widehat{f}(\lambda) \neq 0\}$	p. 35
α_f	The separation constant of an almost periodic function f (where defined)	p. 35
$\\| \cdot \\|_{B^{p,\infty}}$	The weak Besicovitch quasi-norm	p. 38
μ	For an appropriate set E, $\mu(E) \coloneqq \lim_{T\to\infty} \frac{1}{2T} \lvert E \cap [-T,T] \rvert$	p. 39
	For an interval J and $\alpha \in \mathbb{R}^+$:	
$c(J)$	The centre of J	p. 22
αJ	The set $\{\alpha x : x \in J\}$	p. 22
$\alpha \star J$	The interval with the same centre as J and $\lvert \alpha \star J \rvert = \alpha \lvert J \rvert$	p. 22
	For $k \in \mathbb{Z}$, $\rho \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$:	
$\mathbb{D}_k$	**Section 1.2 only:** The collection of all dyadic tiles at a single scale: $\{s \in \mathbb{D} : \lvert I_s \rvert = 2^k\}$	p. 22
$\mathbb{D}_{\rho,\alpha}$	The collection of all α-rescaled dyadic tiles localised to the time window associated to ρ	p. 50
$\mathbb{D}_{\rho,\alpha,k}$	The collection $\mathbb{D}_{\rho,\alpha}$ restricted to a single scale: $\mathbb{D}_{\rho,\alpha,k} \coloneqq \{s \in \mathbb{D}_{\rho,\alpha} : \lvert I_s \rvert = \frac{2^k}{\alpha}\}$	p. 50
	For tiles s and s':	
I_s, ω_s	The time and frequency projections of s in the time-frequency plane: $s = I_s \times \omega_s$	p. 22
ω_s^+, ω_s^-	The upper and lower halves of the frequency projection of s	p. 22
$s < s'$	$I_s \subseteq I_{s'}$ and $\omega_{s'} \subseteq \omega_s$	p. 27
	For an indexed tree $\mathbb{T}_j$:	
I_j, ω_j	The time and frequency projections of the top of $\mathbb{T}_j$: $\mathrm{top}(\mathbb{T}_j) = I_j \times \omega_j$	p. 94
	For a trigonometric polynomial f:	
Λ_f	Representatives in $[0, \alpha_f)$ of the classes of elements in $\sigma(f)$ congruent modulo α_f	p. 51
	For a function f and $a \in \mathbb{R}$:	
$M_a f$	The modulation operator: $M_a f \coloneqq f e^{2\pi i a \cdot}$	p. 24
$\tau_a f$	The translation operator: $\tau_a f \coloneqq f(\cdot - a)$	p. 24
$D_a f$	The dilation operator: $D_a f \coloneqq f(a^{-1} \cdot)$	p. 24

Part II

Boundedness of Maximal Operators of Schrödinger Type

Chapter 5

Introduction to Part II

5.1 The Pointwise Convergence Problem for the Schrödinger Equation

In 1980, a paper by Lennart Carleson entitled "Some Analytic Problems Related to Statistical Mechanics"[46] was published as part of the proceedings of a seminar on harmonic analysis that took place at the University of Maryland in 1979. This was not Carleson's first paper on the topic of statistical mechanics, with a five page note on the subject by him appearing two years earlier in the proceedings of a colloquium that took place in Budapest in 1976[45]. It appears that this earlier paper may not have been well received by the statistical physics community with Robert Minlos, a Russian mathematical physicist who works in the area, writing in *Mathematical Reviews* in 1981 very little other than that "the examples discussed in the note are of too simplified a character and, from the point of view of problems of statistical physics, of little interest."[104] The opening of Carleson's 1980 paper suggests that criticisms of this type may have come to his attention before the article went to print, stating the following:

> *Apology. In the following lectures, I shall give some analytic results which derive from my interest in statistical mechanics. I do not claim any new results for applications, and any serious student of statistical mechanics should consult other sources. It is my hope that analysts will find, as I have, that interesting and difficult analytic problems are suggested by this material; and that they will eventually make contributions of real significance in applications.*[46, p. 5]

Whilst it is unclear to what extent Carleson's paper has, thus far, inspired "contributions of real significance in applications", it is certainly true that "interesting and difficult analytic problems" came to arise from the work. Indeed, in the middle of the second section of the paper, in which he discusses coupled harmonic oscillators, he is led to consider the sequences of functions

$$f_n(x) := \int_{\mathbb{R}} \widehat{f}(\xi) e^{i\frac{\xi^2}{4n}} e^{ix\xi}\, d\xi$$

associated to functions $f : \mathbb{R} \to \mathbb{C}$ of compact support in the Hölder class $\mathscr{H}^\alpha$ for $\alpha \in (0,1)$, that is to say functions of compact support such that

$$|f(x+y) - f(x)| \lesssim |y|^\alpha$$

uniformly for all x and y in $\mathbb{R}$. Asking about pointwise convergence of this sequence, he establishes the following:

Theorem (Carleson, 1980) *Fix any $\varepsilon > 0$ and let $f \in \mathscr{H}^{\frac{1}{4}+\varepsilon}$ be compactly supported. Then*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for almost every $x \in \mathbb{R}$. On the other hand, there exists $f \in \mathscr{H}^{\frac{1}{8}-\varepsilon}$ such that

$$\limsup_{n\to\infty} |f_n(x)| = \infty$$

for almost every $x \in \mathbb{R}$.

In a paper that appeared in 1982, two years later, Björn Dahlberg and Carlos Kenig observed that Carleson's theorem could be recast as a statement about convergence of solutions to the free Schrödinger equation in one spatial dimension,

$$i\partial_t u(t,x) = \partial_x^2 u(t,x).$$

Indeed, for temporal parameter $t > 0$ and spatial variable $x \in \mathbb{R}$, define the following operator acting on functions $f \in \mathscr{S}(\mathbb{R})$:

$$S_t f(x) := \int_{\mathbb{R}} \widehat{f}(\xi) e^{it\xi^2} e^{ix\xi} \, d\xi.$$

Then by applying elementary Fourier transform methods, it is clear that $S_t f(x)$ is the solution, $u(t,x)$, to the free Schrödinger equation with initial data $u(0,x) = f(\frac{x}{2\pi})$.* Clearly, the pointwise behaviour of $S_t f$ as $t \to 0$ is essentially equivalent to the pointwise behaviour of Carleson's sequence, f_n, as $n \to \infty$.

Dahlberg and Kenig observed that the only property of compactly supported functions in the Hölder space $\mathscr{H}^{\frac{1}{4}+\varepsilon}$ used by Carleson in the proof of the positive part of his theorem was that any such function is also contained in the L^2 Sobolev space $H^{\frac{1}{4}}(\mathbb{R})$; as such, Carleson's proof extends to $H^{\frac{1}{4}}(\mathbb{R})$ without modification. In addition, their paper provided a counterexample for the almost everywhere convergence problem in $H^s(\mathbb{R})$ when $s < \frac{1}{4}$. This, combined with the deductions of Carleson, allowed them to conclude the following:

Theorem (Carleson, Dahlberg–Kenig, 1982) *The solution to the Schrödinger equation with initial data $f \in H^s(\mathbb{R})$, given by $S_t f(x)$, converges pointwise to $f(x)$ for almost every $x \in \mathbb{R}$ as $t \to 0$ if and only if $s \geqslant \frac{1}{4}$.*

Both Carleson's proof of the positive part of this theorem and Dahlberg and Kenig's proof of the negative part rely on studying the boundedness properties of the maximal operator associated to $S_t f$,

$$S^* f := \sup_{t \in (0,1)} |S_t f|.$$

*This scaling of the function f is to ensure consistency with the established convention within this thesis that the Fourier transform of a function g is defined as $\widehat{g}(\xi) := \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} \, dx$. This rescaling of initial data will be applied throughout this part of the thesis and has been chosen so that the definitions of operators like S_t can remain consistent to certain cited sources in spite of a different scaling in the definition of the Fourier transform applied there. Since reference to the initial value problems associated to the operators considered in this part of the thesis will generally be restricted to introductory material, it is felt by the author that this approach affords the greatest possible clarity to the exposition.

Carleson's argument essentially established that $\|S^* f\|_{L^2([-1,1])} \lesssim \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}$, which, by standard arguments like those mentioned in the introduction to this thesis (the details of which are found, for example, in [55, Thm. 2.2, p. 27]), implies the desired convergence result. Dahlberg and Kenig fix an even function, $g \in C^\infty(\mathbb{R})$, supported in $[-1,1]$, and for each $v \in (0,1)$, they define the sequence of functions $\widehat{f}_v(\xi) = v\widehat{f}(v\xi + \frac{2}{v})$. This sequence of functions is such that $\|f_v\|_{H^s(\mathbb{R})} \to 0$ as $v \to 0$ for any $s < \frac{1}{4}$. By bounding $|S^* f(x)|$ below by $|S_{v^2 x} f(x)|$ for each $x \in \mathbb{R}$, they show that $\|S^* f\|_{L^{2,\infty}(I)}$ is bounded below, uniformly in v, for any interval I, and thus a bound of the form $\|S^* f\|_{L^{2,\infty}(I)} \lesssim \|f\|_{H^s(\mathbb{R})}$ that is uniform in v is impossible for any $s < \frac{1}{4}$. Invoking the Nikishin maximal principle, about which more will be said in Section 5.2, they conclude that this failure of weak boundedness of the maximal operator shows that the pointwise almost everywhere convergence result fails in $H^s(\mathbb{R})$ for any $s < \frac{1}{4}$.

The counterexample of Dahlberg and Kenig will be significant in the remainder of this part of the thesis, forming the basis for counterexamples in proofs of more general results.

The results of Carleson and Dahlberg and Kenig together give a "complete" picture for the convergence problem in one spatial dimension. A natural generalisation is to consider what other bounds the maximal operator might satisfy. Perhaps the most apparent alternative problem of this type is the question of boundedness of S^* as an operator mapping from $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$, that is to say the problem considered by Carleson but with a global L^2 norm instead of a local one. It follows as a consequence of some other work related to the Schrödinger maximal operator by Luis Vega in his doctoral thesis from 1988[143] that

$$\|S_2^* f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for any $s > \frac{1}{2}$ and that this estimate fails for $s < \frac{1}{2}$.* The problem of boundedness for $s = \frac{1}{2}$ is still an open problem today.

*Specifically, the positive part of this result is a consequence of Teorema 1.15 on p. 35 (by observing that for $t \in (0,1)$ and $\beta > \frac{1}{2}$, the operator S_2^t applied to the function $(\widehat{f}|\cdot|^\beta)^\vee$ is bounded above by $\tau_{2,\beta,t} f$), whilst the negative part is a consequence of part (ii) of Section 2D from the first chapter of the thesis on pp. 44–45.

It is also pertinent to consider the problem in more than one spatial dimension. For fixed $d \in \mathbb{N}$, spatial variable $x \in \mathbb{R}^d$ and temporal variable $t \in \mathbb{R}^+$, the solution to the Schrödinger equation,

$$i\partial_t u(t,x) = \Delta_x u(t,x),$$

with initial data $u(0,x) = f(\frac{x}{2\pi})$, is given by the operator

$$S_t f(x) := \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^2} e^{ix\cdot\xi}\, d\xi.$$

By the multiplicative structure of the multiplier associated to this operator, Carleson's argument extends to higher dimensions, establishing that the maximal operator, S^*, satisfies the bound $\|S^* f\|_{L^2(\mathbb{B}^d)} \lesssim \|f\|_{H^{\frac{d}{4}}(\mathbb{R}^d)}$, where $\mathbb{B}^d$ is the unit ball in $\mathbb{R}^d$. The counterexample of Dahlberg and Kenig can also be used to show that an estimate of the form $\|S^* f\|_{L^2(\mathbb{B}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)}$ cannot hold for any $s < \frac{1}{4}$.

By means of an abstract result on the boundedness of certain maximal operators on Hilbert spaces, Michael Cowling proved in 1983[50] that $\lim_{t\to 0} S_t f(x) = f(x)$ for almost every $x \in \mathbb{R}^d$ whenever $f \in H^s(\mathbb{R}^d)$ if $s > 1$. This result was also obtained by Anthony Carbery in 1985[43] as a consequence of work on maximal operators with radial multipliers, establishing the global bound $\|S^* f\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)}$ for $s > 1$.*

Per Sjölin and Luis Vega, in 1987[129] and 1988[144] respectively, proved independently that the Schrödinger maximal operator satisfies the bound $\|S^* f\|_{L^2(\mathbb{B}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)}$ for any $s > \frac{1}{2}$. Sjölin established this as a consequence of the same result for maximal operators corresponding to solution operators for a class of dispersive partial differential equations that includes the Schrödinger equation (about which more will be said in Chapter 6), whilst Vega proved this as a consequence of a weighted estimate on the Schrödinger maximal operator.†

*Carbery also cites a personal communication in which Elias Stein obtained the same result. He further comments that his result extends to the Sobolev spaces $L^p_s(\mathbb{R}^d)$ for the same choices of s.

†The interested reader may also wish to consult [148] by Kenji Yajima. This paper, published in 1990, shortly after the papers of Sjölin and Vega, establishes this result for maximal operators derived from more general time-dependent Schrödinger equations.

Following Sjölin and Vega's work, there was significant progress on the problem in the case of $d = 2$. In 1992[24], Jean Bourgain proved a slight improvement in this case, showing that there exists $s < \frac{1}{2}$ such that $\|S^* f\|_{L^2(\mathbb{B}^2)} \lesssim \|f\|_{H^s(\mathbb{R}^2)}$, although he was not specific about what the value of s might be, stating that he "[does not] try to optimize the result of the method."[24, p. 1]

Whilst Bourgain's main proof is based on estimating oscillatory integral operators, in an appendix (as well as in [25]), he presents a second approach for establishing the Schrödinger maximal estimate, making use of some of his work on the Fourier restriction problem in $\mathbb{R}^3$. This viewpoint, linking restriction problems and the Schrödinger maximal operator, was to prove fruitful and led to an improvement of Bourgain's result by Adela Moyua, Ana Vargas and Luis Vega in 1996[107]. Whilst Moyua, Vargas and Vega make it clear that their methods provide a better result than simply optimising Bourgain's work would, they too do not explicitly specify the best s attainable from their work, simply stating that there exists $\kappa \in \left(\frac{20}{41}, \frac{41}{84}\right)$ such that $\|S^* f\|_{L^2(\mathbb{B}^2)} \lesssim \|f\|_{H^s(\mathbb{R}^2)}$ for all $s > \kappa$. In fact, the optimal value of κ resulting from their proof is $\frac{164+\sqrt{2}}{339} = 0.4879475\ldots$*. This result is generalised by the same authors in [108], published three years later, to the aforementioned class of maximal operators considered by Sjölin.

The next improvement to the two-dimensional result was by Terence Tao and Ana Vargas in a two part paper[139, 140] published in 2000. They comment in the second part that work from the first part, as well as work from [141] by Tao, Vargas and Vega, could be used to directly refine the method of Moyua, Vargas and Vega. However, they produce a better result using an approach based on bilinear restriction estimates, proving that $\lim_{t\to 0} S_t f(x) = f(x)$ for almost every $x \in \mathbb{R}^2$ whenever $f \in H^s(\mathbb{R}^2)$ if $s > \frac{15}{32} = 0.46875$. This is shown as a consequence of bounds of the form $\|S^* f\|_{L^{2q}(\mathbb{R}^2)} \lesssim \|f\|_{H^s(\mathbb{R}^2)}$ which are shown to hold when $s > 1 - \frac{1}{q}$ and q is an index that satisfies a certain restriction problem. As it is shown that q may be taken to be any real number larger than $\frac{32}{17}$, the lower bound of $\frac{15}{32}$ for s follows. Tao and Vargas comment

*This can be seen substituting $p = \frac{7}{3}$, $q = \frac{42}{11}$ and $r = \frac{4}{1+\sqrt{2}}$ into the bound $\kappa \leqslant \frac{1}{2} - \dfrac{\frac{1}{q} - \frac{1}{4}}{\left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{4-r}{2-r}\right) + \frac{1}{2}}$ given on p. 813.

that the lowest feasible value for q is $\frac{5}{3}$ and thus that the smallest possible boundary value for s that can be obtained from this method is $\frac{2}{5}$. This value was obtained in this way by Tao alone in 2003[138], three years later. Further, in this later paper, Tao provided restriction results in dimensions higher than 2, allowing the argument from [140] to be applied to the convergence problem for all $d \geqslant 2$, showing that $\lim_{t\to 0} S_t f(x) = f(x)$ for almost every $x \in \mathbb{R}^d$ whenever $f \in H^s(\mathbb{R}^d)$ if $s > \frac{d}{d+3}$, although this is a higher boundary value for s than that obtained by Sjölin and Vega in the case of $d \geqslant 4$ and is equal to it for $d = 3$.

The current best result in two dimensions was established by Sanghyuk Lee in 2006[95]. Whilst Lee's proof was also based on bilinear restriction estimates, instead of using restriction results directly as had been done previously, he adapted the arguments used in their proofs to the context of the Schrödinger maximal bound, proving that the $s > \frac{2}{5}$ condition of Tao could be improved to $s > \frac{3}{8}$.

For $d \geqslant 3$, the result of Sjölin and Vega establishing boundedness for $s > \frac{1}{2}$ remains the best published convergence result to date. Nonetheless, a pre-print by Bourgain[23] appeared on `arXiv.org` in January 2012 in which it is claimed that for any $d \in \mathbb{N}$, $\lim_{t\to 0} S_t f(x) = f(x)$ for almost every $x \in \mathbb{R}^d$ whenever $f \in H^s(\mathbb{R}^d)$ if $s > \frac{1}{2} - \frac{1}{4d}$. This recovers the result of Carleson for $d = 1$ and the result of Lee for $d = 2$. Bourgain's proof is based on his previous work with Larry Guth in [27] which is a development of the multilinear restriction theory developed by Jonathan Bennett, Anthony Carbery and Terence Tao in [16]. Further, in the same paper, Bourgain claims the first improvement of the counterexample of Dahlberg and Kenig, establishing that for $d \geqslant 4$, the convergence result can only hold on $H^s(\mathbb{R}^d)$ if $s \geqslant \frac{1}{2} - \frac{1}{d}$.

Of course, in addition to the results mentioned here, there are many other natural problems of this form. In a paper published in 2011, Juan Antonio Barceló, Jonathan Bennett, Anthony Carbery and Keith Rogers[14] considered the problem of quantifying further the size of the sets on which solutions to the Schrödinger equation can fail to converge to the initial data by determining their Hausdorff dimension, following on from previous work of Peter Sjögren and Per Sjölin[127] from 1989 carried out in terms of Sobolev capacities. This question requires formulating delicately since equivalent L^2 functions can differ on a Lebesgue null set of full Hausdorff dimension. Also, in addition to the aforementioned result of Tao, Vargas and Vega, boundedness of Schrödinger maximal operators from H^s to L^p for $p \neq 2$ has been considered by a number of authors, including boundedness of the Schrödinger maximal operator with a global supremum in time. The latest results of this type and references to earlier works can be found in [120].

5.2 The Maximal Principles of Stein, Sawyer and Nikishin

As has already been discussed briefly in the introduction to this thesis and Section 1.1, whilst pointwise convergence results for sequences of operators, $(T_n)_{n\in\mathbb{N}}$, acting on L^p follow from boundedness of the maximal operator $T^* \coloneqq \sup_{n\in\mathbb{N}} |T_n \cdot|$ as a map from L^p to $L^{q,\infty}$ for some $q \in [1,\infty]$, it is also the case in the appropriate setting that the problems of pointwise convergence and boundedness of maximal operators are equivalent to one another. Results of this form will be useful for proving divergence results in the remainder of this thesis, as was the case in the aforementioned paper of Dahlberg and Kenig[51]. In the literature on maximal principles, theorems are usually either not applicable to the operator bounds considered here or stated very generally with the application to the present setting unclear. As such, the purpose of this section is to provide an overview of the landmark maximal principles of the twentieth century, ultimately leading to Nikishin's maximal principle and discussing its applicability to results like that proved by Dahlberg and Kenig.

For further information on this topic, the reader is referred to [71] as well as the references given throughout this section. There are also discussions of the results of Banach, Stein and Sawyer in [63, Ch. 1] and of the "factorisation" framework behind Nikishin's maximal principle (illustrated with a number of examples in pointwise convergence problems) in [62, Sec. 6.2].

Early results on pointwise convergence and maximal operators were given by Banach in 1926[13]. Their statement will require the following definition, as stated by Banach:

Definition (Continuity in Measure) *A linear operator T, mapping from a Banach space X to $\mathscr{M}([0,1])$, is said to be continuous in measure if for any $\lambda > 0$, $|\{x \in [0,1] : Tf(x) > \lambda\}| \to 0$ as $\|f\|_X \to 0$.*

Recall that the symbol $\mathscr{M}([0,1])$ is used in this thesis to represent the space of all Lebesgue measurable functions mapping from $[0,1]$ to $\mathbb{C}$. More generally, for a suitable measure space (Ω, Σ, μ), the symbol $\mathscr{M}(\Omega, \mu)$ is used to represent the space of all μ-measurable functions mapping from Ω to $\mathbb{C}$.

One of the results of Banach can now be stated as follows:[13, Thm. 1]

Theorem (Banach, 1926) *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of linear operators that are continuous in measure and map from a Banach space X to $\mathscr{M}([0,1])$. If for each $f \in X$, $\limsup_{n\to\infty} T_n f(x)$ is finite for almost every $x \in [0,1]$, then the maximal operator $T^* := \sup_{n\in\mathbb{N}} |T_n \cdot|$ is continuous in measure.*

This theorem immediately establishes a qualitative property (that is, continuity in measure) of the maximal operator, T^*, as a consequence of convergence almost everywhere of $T_n f$ for each f. Nonetheless, later in Banach's paper, it has a more profound consequence in the way of establishing a result of reciprocal form, that is using a qualitative property of T^* to deduce the convergence almost everywhere of $T_n f$ for each f. Indeed, Banach deduces the following from the above theorem:[13, Thm. 3]

Theorem (Banach, 1926) *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of linear operators that are continuous in measure and map from a Banach space X to $\mathscr{M}([0,1])$. Suppose that for each $f \in X$, $\limsup_{n\to\infty} T_n f(x) < \infty$ for almost every $x \in [0,1]$ and that for each f in some dense subspace of X, $\lim_{n\to\infty} T_n f$ exists almost everywhere. Then for* all *$f \in X$, $\lim_{n\to\infty} T_n f$ exists almost everywhere.*

Since the finiteness almost everywhere of $\limsup_{n\to\infty} T_n f(x)$ is equivalent to the finiteness almost everywhere of $T^* f(x)$ in this context (given that the T_n map onto a space of functions that are finite almost everywhere), it follows that a pointwise convergence result can be deduced from the finiteness almost everywhere of the maximal operator. It is an *a priori* stronger assumption to require some form of weak boundedness for T^*, as mentioned at the beginning of this section (although this assumption allows for a more straightforward proof of the convergence result; the reader is once again referred to [55, Thm. 2.2, p. 27]). Nonetheless, results subsequent to Banach's work strongly suggest that weak boundedness is the "correct" perspective on maximal operators in relation to pointwise convergence problems. As was already mentioned in Section 1.1, Antoni Zygmund communicated a result of Alberto Calderón in 1959 showing that pointwise almost everywhere convergence of partial sums of Fourier series for all functions in $L^2(\mathbb{T})$ implies that the maximal Fourier partial sum operator is bounded as a map from $L^2(\mathbb{T})$ to $L^{2,\infty}(\mathbb{T})$.[149, Thm. XIII.1.22, p. 165] It was also mentioned in Section 1.1 that this result was generalised by Stein to a wider setting in [135], published in 1961. The original scope of Stein's theorem is as follows:

Theorem (Stein, 1961) *Let G be a compact group and let X be a space upon which G acts transitively equipped with a G-invariant measure.** *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear operators that map $L^p(X)$ into itself for some $p \in [1,2]$ and commute with the group action from G.*

Suppose that for all $f \in L^p(X)$, $\lim_{n\to\infty} T_n f(x) < \infty$ for almost every $x \in X$. Then T^* is bounded as a map from $L^p(X)$ to $L^{p,\infty}(X)$.

In a standard Euclidean context (for example, $X = \mathbb{T}$), the group G here should be thought of as the group of translations and thus the T_n are operators that are assumed to commute with translations. It is a well known fact that bounded linear operator on Euclidean L^p spaces that commute with translations are necessarily of convolution form (see, for example, [66, Thm. 2.5.2]), so the scope of this theorem in this setting is convolution operators.

Stein comments that the assumption that $\lim_{n\to\infty} T_n f(x) < \infty$ may be replaced with the weaker assumption that $\limsup_{n\to\infty} |T_n f(x)| < \infty$ for x in *some set of positive measure* (it may hence also be replaced with the assumption that $|T^* f(x)| < \infty$ for x in some set of positive measure†). He also notes that whilst his result is formulated for a countable family of operators, $(T_n)_{n\in\mathbb{N}}$,

*In Stein's language[135, Sec. II.7], X is said to be a homogeneous space of G.

†It follows that in this setting, finiteness and weak boundedness of the maximal operator are equivalent.

it can equally be applied to families depending on a continuous parameter, $(T_t)_{t\in E}$ for some uncountable set $E \subseteq \mathbb{R}$, by observing that

$$\sup_{t\in E}|T_t \cdot| = \sup_{t\in E\cap\mathbb{Q}} |T_t \cdot|.$$

In spite of the abstract setting, there are some significant restrictions in the applicability of this theorem. Firstly, the restriction that G be compact prohibits working with operators mapping $L^p(\mathbb{R})$ into itself, for example. This is perhaps not surprising, given the discrepancy between the regularity required for local and global bounds on the Schrödinger maximal operator, as discussed in Section 5.1. Of course, this example raises the additional question of validity of results of this form in the setting of operators mapping from global spaces into local spaces, since the bounds on the Schrödinger maximal operator from Section 5.1 were in the context of maps from $H^s(\mathbb{R})$ into $L^2([-1,1])$. Finally, the restriction that $p \in [1,2]$ is obviously a significant one.

As Stein comments, and demonstrates by means of a counterexample[135, Sec. 12], this latter condition is necessary for the result as it is formulated. Nonetheless, in an appendix, he does remark that the proof of the theorem has sufficient scope to generalise the result to the setting where the operators T_n are bounded as maps from a space B into itself, where B is chosen from a wide class of Banach spaces of functions on X (including the spaces $L^p(X)$ for all choices of $p \in [1,\infty]$), concluding boundedness of T^* as a map from B to $L^{1,\infty}(X)$ as a consequence of pointwise convergence. He also provides the additional details required to generalise this conclusion further to boundedness of T^* into $L^{2,\infty}(X)$ when $B = L^p(X)$ for some $p \in [2,\infty)$.

An alternative generalisation of Stein's theorem that removes the condition "$p \in [1,2]$" was given by Stanley Sawyer in a paper based on work from his doctoral thesis[124] and published in *The Annals of Mathematics* in 1966[123], obtaining boundedness of the maximal operator from L^p to $L^{p,\infty}$ at the expense of requiring non-negativity of the operators (that is to say that non-negativity of functions is preserved by the operators under consideration). The statement of this result requires the following definition:

Definition (Ergodic Family of Operators) *Let (X, Σ, μ) be a probability space. A collection $\mathscr{F}$ of measure-preserving transformations of X is said to be an "ergodic family" if for any $A \in \Sigma$ for which $F^{-1}(A) = A$ for all $F \in \mathscr{F}$ up to μ-null sets, it is necessarily the case that A has either full or zero measure.*

It is remarked that this condition is significantly weaker than requiring that all members of $\mathscr{F}$ are ergodic.

Sawyer's theorem[123, Thm. 2] may now be stated as follows:

Theorem (Sawyer, 1966) *Fix $p \in [1, \infty)$ and let (X, Σ, μ) be a probability space. Suppose that $(T_n)_{n\in\mathbb{N}}$ is a sequence of non-negative linear operators mapping $L^p(X)$ into $\mathscr{M}(X, \mu)$ and that there is some ergodic family of measure-preserving transformations, $\mathscr{F}$, mapping X to X such that for all $S \in \mathscr{F}$ and for each $f \in L^p(X)$, $T^* f(Sx) \leqslant T^* f(x)$ for almost every $x \in X$. Then if for every $f \in L^p(X)$, $T^* f < \infty$ almost everywhere, T^* is bounded as a map from $L^p(X)$ into $L^{p,\infty}(X)$.*

If for all $f \in L^p(X)$, $\lim_{n\to\infty} T_n f(x) < \infty$ for almost every $x \in X$, then the hypothesis that $T^* f < \infty$ almost everywhere follows automatically here, as mentioned before. Note that Sawyer refers to the property that $T^* f(Sx) \leqslant T^* f(x)$ as the sequence $(T_n)_{n\in\mathbb{N}}$ "commuting with S".

Sawyer comments that his setting is more general than that of Stein in that the requirement of commutativity with measure-preserving transformations from an ergodic family is weaker than Stein's requirement of commutativity with a transitive group action, and prior to the above theorem he also states in [123] a theorem that is simply this generalisation of Stein's result, retaining the requirement that $p \in [1, 2]$ and not requiring non-negativity of the operators T_n. Like Stein, he also proves a result in a more abstract setting than L^p spaces, although his generalisation is slightly different.[123, Thm. 3]

The most important generalisations of Stein's result in the context of this thesis are results due to Nikishin, dating from the 1970s. Unlike the results considered so far in this section, the scope of these theorems goes beyond the realm of maximal operators. Their statement will require the following definition, as given as Definition 4 in [109]:

Definition (Superlinear Operators) *An operator T mapping from some Banach space B to $\mathscr{M}([0,1])$ is said to be superlinear in the sense of Nikishin* * *if for any $f \in B$, there exists a linear operator T_f mapping from B into $\mathscr{M}([0,1])$ such that $T_f f = Tf$ and for any $g \in B$, $|T_f g(x)| \leqslant |Tg(x)|$ for almost every $x \in [0,1]$.*

It is remarked that maximal operators of the form $T^* f(x) := \sup_{1 \leqslant n \leqslant N} |T_n f(x)|$ for some $N \in \mathbb{N}$ satisfy this definition. This can be seen by observing that for each input function, f, there exists a function n_f mapping from the domain of f into $[1,N] \cap \mathbb{N}$ such that $T^* f(x) = |T_{n_f(x)} f(x)|$. In the case where the limit $\lim_{n\to\infty} T_n \cdot$ is known to exist, N may be infinite, n_f then being allowed to take the value ∞ with T_∞ defined to correspond to this limit.

In [109], Nikishin proves a number of results on superlinear operators, the most relevant here being the following[109, Thm. 2]:

Theorem (Nikishin, 1970) *Let (X,μ) be a σ-finite measure space, fix $p \in [1,\infty)$ and let T be a superlinear operator mapping from $L^p(X,\mu)$ into $\mathscr{M}([0,1])$ that is continuous in measure*†. *Then for any $\varepsilon > 0$ and $q \in (0, \min(p,2))$, there exists a set $E \subseteq [0,1]$ with $|E| \geqslant 1-\varepsilon$ such that*

$$\|Tf\|_{L^q(E)} \lesssim_{\varepsilon,q} \|f\|_{L^p(X,\mu)}$$

for all $f \in L^p(X,\mu)$.

In the context of the problem at hand, in this theorem, "T" should be thought of as a maximal operator T^* corresponding to a sequence of operators, $(T_n)_{n\in\mathbb{N}}$, where $\lim_{n\to\infty} T_n f(x)$ exists almost everywhere for all $f \in L^p(X,\mu)$. As mentioned above, the first theorem of Banach can be used to conclude that T^* is continuous in measure and thus an $L^p \to L^q$ boundedness result for T^* can be deduced from the above theorem.

*This term should not be confused with "superlinear" in the sense that $T(f+g) \geqslant Tf + Tg$ for any f, $g \in B$. An alternative appropriate term, as used in [71], is "linearisable".

†The hypothesis given by Nikishin is that T is "bounded" in accordance with his Definition 6. It is easily seen that this is equivalent to the definition of continuity in measure given by Banach.

Nikishin proves this theorem as an intermediate step for establishing results based on Edmund Landau's "resonance" theorem originating from [93].* With respect to Stein's theorem, he simply writes, "Сравнение с результатами Штейна будет сделано в следующих публикациях."[109, p. 133] [A comparison with the results of Stein will be made in later publications.]

In the fifth section of the second chapter of [109], Nikishin considers the possibility of improving the resonance theorems derived from the above theorem. He proves that, at least for these derived results, it is not possible to choose $q = \min(p,2)$ for $p \in [1,2)$. By examining the proof of his theorems, it is clear that the same may consequently be said for the above theorem. He goes on to explain that it is unknown whether q can be increased for $p \in [2,\infty)$, commenting further, "особый интерес представляет случай $p = 2$"[109, p. 176] [the case of $p = 2$ is particularly interesting], in light of its consequences in his applications.

Nikishin managed to address this "interesting" case in [114]. This paper also seems to be the first "later publication" in which he directly compared his results with those of Stein. In it he establishes the following theorem[114, Thm. 2], which had also appeared previously as Theorem 2 in [112], an abridged publication of his higher (that is, second) doctoral thesis[111]:

Theorem (Nikishin, 1971) *Let (X,μ) be a σ-finite measure space, fix $p \in [1,\infty)$ and let T be a superlinear operator mapping from $L^p(X,\mu)$ into $\mathcal{M}([0,1])$ that is continuous in measure. Then for any $\varepsilon > 0$, there exists a set $E \subseteq [0,1]$ with $|E| \geqslant 1-\varepsilon$ such that*

$$\|Tf\|_{L^{\min(p,2),\infty}(E)} \lesssim_\varepsilon \|f\|_{L^p(X,\mu)}$$

for all $f \in L^p(X,\mu)$.

*Landau's theorem states that if for a sequence $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and some fixed $p > 1$, $\sum_{n\in\mathbb{N}} a_n b_n$ converges for every $(a_n)_{n\in\mathbb{N}} \in \ell^p$, then necessarily $(b_n)_{n\in\mathbb{N}} \in \ell^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Landau does not use the term "resonance" himself.

As in the aforementioned appendix of Stein, in both [114] and [112], as Theorem 3 and Theorem 1 respectively, Nikishin also states a result for superlinear operators T acting on functions from a Banach space B concluding boundedness of T as a map from B into $L^{1,\infty}(E)$.

In the context of the Schrödinger maximal operator and the other maximal operators considered in this part of the present thesis, the usefulness of these two theorems of Nikishin is clouded by the presence of the set E in the statements, which may not be equal to $[0,1]$. If the operator T were assumed to be translation-invariant and E could be assumed to be open, the fact that finitely many translates of E cover $[0,1]$ (by compactness) could be used to formulate a version of these theorems with E replaced with $[0,1]$. However, whilst the assumption that T is translation-invariant is a reasonable hypothesis, E cannot be assumed to be open, and it is not possible in general to cover $[0,1]$ with finitely many translates of an arbitrary set of less than full measure, even up to a null set.* Nonetheless, given that the measure of E may be chosen to be as close to 1 as is desired, the fact that E cannot be forced to be equal to $[0,1]$ is ultimately not a significant obstacle when one wishes to show that pointwise convergence fails as a result of failure of boundedness of a maximal operator. Indeed, when Dahlberg and Kenig apply the second of the two theorems by Nikishin above in [51] to conclude failure of the pointwise convergence result from failure of weak boundedness of the Schrödinger maximal operator, they do so by assuming convergence holds and then exhibiting a counterexample for weak boundedness on the set E deduced from Nikishin's theorem. In general, this is not generally significantly more difficult than exhibiting a counterexample on a set of full measure.

Whilst the problems considered in this thesis and the setting of Dahlberg and Kenig are not included in it, there is a class of problems for which Nikishin's results may be stated with $E = [0,1]$. Indeed, on p. 169 of [109] (translated on pp. 164-165 of [110]), Nikishin considers boundedness of the Hardy–Littlewood maximal operator and manages to establish that within the context of the bounds he is considering, the conclusions of his theorems are

*This can be demonstrated using the so-called Smith–Volterra–Cantor set (see for example [41, Ch. 4]) which is a set of positive measure that is defined in the same way as a regular Cantor set, but with a smaller proportion of the set removed at each stage of the construction.

valid with $E = [0,1]$. This argument generalises without much difficulty to other operators; the details of this generalisation will be presented here in the form of corollaries of the two theorems above with more restrictive hypotheses but in which the set E is replaced with $[0,1]$. To begin with, the following simple lemma, found, for example, on p. 17 of [79] (and there attributed to Fejér), is needed:

Lemma *Suppose that $f \in L^\infty([0,1])$ and $g \in L^1([0,1])$, with g identified with a periodic function on $\mathbb{R}$. Then*

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)\,dx = \Big(\int_0^1 f(x)\,dx\Big)\Big(\int_0^1 g(x)\,dx\Big).$$

This can be proved by identifying both f and g with periodic functions on $\mathbb{R}$ and approximating them by trigonometric polynomials.

The first theorem of Nikishin stated above has the following corollary:

Corollary 5.2.1 *Fix $p \in [1,\infty)$, let X be a complex vectorspace and let (X,μ) be a σ-finite measure space. Suppose that for any $n \in \mathbb{N}$ and any $f \in L^p(X,\mu)$, $\|f(n\cdot)\|_{L^p(X,\mu)} \lesssim \|f\|_{L^p(X,\mu)}$ uniformly in n. Further, let T be a superlinear operator mapping from $L^p(X,\mu)$ into $\mathscr{M}([0,1])$* that is continuous in measure and is such that for any $n \in \mathbb{N}$ and any $f \in L^p(X,\mu)$, $|(Tf)(n\cdot)| \lesssim |T(f(n\cdot))|$ for almost every $x \in [0,1]$, uniformly in n. Then for any $q \in (0,\min(p,2))$*

$$\|Tf\|_{L^q([0,1])} \lesssim_q \|f\|_{L^p(X,\mu)}$$

for all $f \in L^p(X,\mu)$.

Proof Fix $\varepsilon > 0$ and choose any $f \in L^p(X,\mu)$. Then by the first theorem of Nikishin, there exists a set $E \subseteq [0,1]$ with $|E| \geqslant 1-\varepsilon$ such that $\|Tg\|_{L^q(E)} \lesssim_{\varepsilon,q} \|g\|_{L^p(X,\mu)}$ for any $g \in L^p(X,\mu)$. Consequently, for any $n \in \mathbb{N}$,

$$\|T(f(n\cdot))\|_{L^q(E)} \lesssim_{\varepsilon,q} \|f(n\cdot)\|_{L^p(X,\mu)} \lesssim \|f\|_{L^p(X,\mu)}.$$

*Here and throughout the rest of this section, functions on $[0,1]$ will continue to be identified with periodic functions on $\mathbb{R}$.

Using that $|(Tf)(n\cdot)| \lesssim |T(f(n\cdot))|$ for almost every $x \in [0,1]$, as well as that $\|f(n\cdot)\|_{L^p(X,\mu)} \lesssim \|f\|_{L^p(X,\mu)}$, it follows that

$$\|(Tf)(n\cdot)\|_{L^q(E)} \lesssim_{\varepsilon,q} \|f\|_{L^p(X,\mu)}.$$

However, using the lemma of Fejér from above,

$$\begin{aligned}\lim_{n\to\infty} \|(Tf)(n\cdot)\|^q_{L^q(E)} &= \lim_{n\to\infty} \int_0^1 \chi_E(x)|(Tf)(nx)|^q \, dx \\ &= |E| \int_0^1 |(Tf)(x)|^q \, dx.\end{aligned}$$

It thus follows that

$$\|Tf\|_{L^q([0,1])} \lesssim_{\varepsilon,q} \|f\|_{L^p(X,\mu)},$$

as required. □

Note that this method can immediately be adapted to the setting of the second theorem of Nikishin by observing that

$$\begin{aligned}|\{x \in E : |(Tf)(nx)| > \lambda\}| &= \int_0^1 \chi_E(x)\chi_{\{x\in[0,1]:|(Tf)(nx)|>\lambda\}}(x) \, dx \\ &= \int_0^1 \chi_E(x)\chi_{\{x\in[0,1]:|Tf(x)|>\lambda\}}(nx) \, dx,\end{aligned}$$

where the second characteristic function in the second integral is treated as 1-periodic. It can thus be seen that the following result also holds:

Corollary 5.2.2 *Fix $p \in [1,\infty)$, let X be a complex vectorspace and let (X,μ) be a σ-finite measure space. Suppose that for any $n \in \mathbb{N}$ and any $f \in L^p(X,\mu)$, $\|f(n\cdot)\|_{L^p(X,\mu)} \lesssim \|f\|_{L^p(X,\mu)}$ uniformly in n. Further, let T be a superlinear operator mapping from $L^p(X,\mu)$ into the space of Lebesgue measurable functions on $[0,1]$ that is continuous in measure and is such that for* *any* $n \in \mathbb{N}$ *and any* $f \in L^p(X,\mu)$, $|(Tf)(n\cdot)| \lesssim |T(f(n\cdot))|$ *for almost every* $x \in [0,1]$, *uniformly in* n. *Then*

$$\|Tf\|_{L^{\min(p,2),\infty}([0,1])} \lesssim \|f\|_{L^p(X,\mu)}$$

for all $f \in L^p(X,\mu)$.

The reason that Corollaries 5.2.1 and 5.2.2 cannot be applied to the Schrödinger maximal operator, for example, is that the condition that $|(Tf)(n\cdot)| \lesssim |T(f(n\cdot))|$ for almost every $x \in [0,1]$ uniformly across $n \in \mathbb{N}$ cannot be verified, given the required periodisation of the operator T.

Chapter 6

Maximal Operators Associated to Dispersive Equations with Complex Time

Note: *the work presented in this chapter has previously appeared in the author's paper, [9].*

6.1 Introduction

In this chapter, the boundedness of maximal operators associated to a natural class of dispersive partial differential equations that includes the Schrödinger equation will be considered. For $a > 1$, temporal parameter $t > 0$ and spatial variable $x \in \mathbb{R}$, define the following operator acting on functions $f \in \mathscr{S}(\mathbb{R})$:

$$S_a^t f(x) := \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{ix\xi} \, d\xi.$$

This operator gives the solution to the dispersive equation

$$i\partial_t u(t,x) + (-\Delta)^{\frac{a}{2}} u(t,x) = 0$$

in one spatial dimension with initial data $u(0,x) = f(\frac{x}{2\pi})$. Here, the operator $(-\Delta)^{\frac{a}{2}}$ is defined in terms of its multiplier representation,

$$((-\Delta)^{\frac{a}{2}}\phi)\widehat{\ }(\xi) := (2\pi|\xi|)^a \widehat{\phi}(\xi)$$

for $\phi \in \mathscr{S}(\mathbb{R})$. This definition is motivated by the fact that the Laplacian on $\mathbb{R}^d$ with $d \in \mathbb{N}$ satisfies $\widehat{-\Delta\phi}(\xi) = 4\pi^2|\xi|^2\widehat{\phi}(\xi)$ for any $\phi \in \mathscr{S}(\mathbb{R}^d)$. For $a = 2$, S_a^t corresponds to the solution operator for the Schrödinger equation.

It is a simple observation that if the definition of the solution operator for the Schrödinger equation is extended to allow complex-valued time with positive imaginary part, then for $t \geqslant 0$, the operator S_2^{it} is the solution operator for the heat equation,

$$\partial_t u(t,x) = \partial_x^2 u(t,x).$$

Here, the solution operator, S_2^{it}, is a convolution operator with a Gaussian multiplier and hence also a Gaussian kernel. It is thus standard (see, for example, [66, Thm. 2.1.10, p. 82]) that $\sup_{t\in(0,1)} |S_2^{it} f(x)|$ can be pointwise dominated by $Mf(x)$ where M is the Hardy–Littlewood maximal operator. As such, by boundedness of M,

$$\Big\| \sup_{t\in(0,1)} |S_2^{it} f| \Big\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}.$$

Given this result and the result of Vega[143] on global boundedness of the Schrödinger maximal operator which established that

$$\| \sup_{t\in(0,1)} |S_2^t f| \|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

if $s > \frac{1}{2}$ and only if $s \geqslant \frac{1}{2}$ (as mentioned in Section 5.1), it is natural to consider intermediate results, asking for which $s > 0$ and which maps $g : [0,1] \to [0,1]$ with $\lim_{t\to 0} g(t) = 0$ it can be said that

$$\Big\| \sup_{t\in(0,1)} |S_2^{t+ig(t)} f| \Big\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}.$$

Of course, given the work of Carleson, Dahlberg and Kenig discussed in Section 5.1, it is also natural to consider an analogous local boundedness question (that is with the $L^2(\mathbb{R})$ norm replaced with a norm on $L^2([-1,1])$). Remarkably, in this setting, it turns out that an answer to this question can be determined as a rather straightforward consequence of the methods employed herein to consider the global question. As such, its resolution will be delayed until Section 6.4.

The global complex time question was posed and partially answered by Per Sjölin in [132]. For $t, \gamma > 0$, he considered the operator

$$P^t_{2,\gamma} f(x) \coloneqq S_2^{t+it^\gamma} f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{it\xi^2} e^{-t^\gamma \xi^2} e^{ix\xi}\, d\xi$$

with corresponding maximal operator $P^*_{2,\gamma} f \coloneqq \sup_{t\in(0,1)} |P^t_{2,\gamma} f|$. Denoting by $s_2(\gamma)$ the infimum of the values of $s > 0$ such that the estimate $\|P^*_{2,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ holds, the following is established by Sjölin in [132] and by Sjölin together with Fernando Soria in [133]:

Theorem 6.1.1

(i) *For* $\gamma \in (0,1]$, $s_2(\gamma) = 0$;
(ii) *For* $\gamma \in (1,2)$, $s_2(\gamma) \in [0, \frac{1}{2} - \frac{1}{2\gamma}]$;
(iii) $s_2(2) = \frac{1}{4}$;
(iv) *For* $\gamma \in (2,4]$, $s_2(\gamma) \in [\frac{1}{4}, \frac{1}{2} - \frac{1}{2\gamma}]$;
(v) *For* $\gamma \in (4,\infty)$, $s_2(\gamma) \in [\frac{1}{2} - \frac{1}{\gamma}, \frac{1}{2} - \frac{1}{2\gamma}]$.

Sjölin's original work in [132] established cases (i) and (iii), as well as (v) with an upper bound of $\frac{1}{2}$ instead of $\frac{1}{2} - \frac{1}{2\gamma}$. Cases (ii) and (iv) and the improvement of case (v) were established in [133] with Soria.

In a paper from 1994[130], Sjölin also established that for any $a > 1$, the infimum of the values of $s > 0$ for which

$$\|S^*_a f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

holds is $\frac{a}{4}$. As such, in addition to the problem of fully determining $s_2(\gamma)$ in cases (ii), (iv) and (v), it is natural to consider whether the scope of Theorem 6.1.1 can be extended to $a \neq 2$. To this end, for $t, \gamma > 0$ and $a > 1$, define

$$P^t_{a,\gamma} f(x) \coloneqq S^{t+it^\gamma}_a f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{-t^\gamma|\xi|^a} e^{ix\xi}\, d\xi$$

with corresponding maximal operator $P^*_{a,\gamma} f \coloneqq \sup_{t\in(0,1)} |P^t_{a,\gamma} f|$ and let $s_a(\gamma)$ denote the infimum of the values of $s > 0$ such that the estimate $\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ holds. Letting x^+ denote the maximum of 0 and x for each $x \in \mathbb{R}$, the following result is established in this chapter:

Theorem 6.1.2 *For $\gamma \in (0,\infty)$ and $a > 1$, $s_a(\gamma) = \frac{1}{4}a\left(1 - \frac{1}{\gamma}\right)^+$.*

Note that in the case of $a = 2$, this "completes" Theorem 6.1.1 of Sjölin and Soria.

In Section 5.2, the link between estimates on maximal operators and pointwise convergence results was discussed in some detail. Here, pointwise almost everywhere convergence of $P^t_{a,\gamma} f$ to f as $t \to 0$ can be deduced as a corollary of Theorem 6.1.2 for functions $f \in H^s(\mathbb{R})$ when a and γ are as in the hypotheses of the theorem and $s > s_a(\gamma)$. In fact, the following stronger result will also be established:

Theorem 6.1.3 *For each $\gamma \in (0,\infty)$ and $a > 1$, the infimum of the values of $s > 0$ for which*

$$\lim_{t\to 0} P^t_{a,\gamma} f(x) = f(x)$$

for almost every $x \in \mathbb{R}$ whenever $f \in H^s(\mathbb{R})$ is $\min\left(\frac{1}{4}a\left(1-\frac{1}{\gamma}\right)^+, \frac{1}{4}\right)$. Moreover, this convergence also occurs for all $f \in L^2(\mathbb{R})$ when $\gamma \in (0,1]$ and for all $f \in H^{\frac{1}{4}}(\mathbb{R})$ when $\gamma \in [\frac{a}{a-1}, \infty)$.

Theorem 6.1.3 will be proved in Section 6.4 as a consequence of the aforementioned local bounds for $P^*_{a,\gamma}$.

To prove Theorem 6.1.2, it will suffice to consider the case of $\gamma > 1$, owing to the following generalisation of Sjölin's Lemma 1 from [132]:

Lemma 6.1.4 *Let g and h be continuous functions mapping $[0,1]$ to $[0,1]$ such that $g(t) \leqslant h(t)$ for all $t \in (0,1)$. Then for any $a > 1$,*

$$\| \sup_{t\in(0,1)} |S_a^{t+ih(t)} f| \|_{L^2(\mathbb{R})} \lesssim \| \sup_{t\in(0,1)} |S_a^{t+ig(t)} f| \|_{L^2(\mathbb{R})}$$

for any $f \in \mathscr{S}(\mathbb{R})$.

In addition to reducing the proof of Theorem 6.1.2 to the case of $\gamma > 1$, this lemma also suggests that in terms of understanding the convergence at the origin, the $P^*_{a,\gamma}$ are natural operators to consider as they encapsulate the convergence properties of any operator of the form $S_a^{t+ih(t)}$ when $h(t)$ is of polynomial type near $t = 0$.

The proof of Lemma 6.1.4 is essentially the same as the proof of the analogous result from [132] and will be postponed until Section 6.5 at the end of this chapter.

The proof of Theorem 6.1.2 will be divided into two sections. In Section 6.2, it will be shown that $\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ holds for all s above the critical index, $\frac{1}{4}a\left(1-\frac{1}{\gamma}\right)$ when $\gamma > 1$, whilst in Section 6.3, it will be shown that this boundedness fails for all s below this index. Section 6.4 will contain some further remarks on the implications of Theorem 6.1.2 and its proof, as well as the proofs of Theorem 6.1.3 and a local boundedness theorem.

6.2 Boundedness for Regularity Above the Critical Index

It is claimed that to show that $\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ for all $s > \frac{1}{4}a\left(1-\frac{1}{\gamma}\right)$ where $\gamma > 1$, it will be sufficient to prove the following lemma:

Lemma 6.2.1 *Suppose that $a, \gamma > 1$ and $\alpha > \frac{1}{2}a\left(1-\frac{1}{\gamma}\right)$. If $\gamma < \frac{a}{a-1}$, suppose further that $\alpha < \frac{1}{2}$. Let $\mu \in \mathscr{S}(\mathbb{R})$ be compactly supported, positive, even, real-valued and radially decreasing. Then there exists $K \in L^1(\mathbb{R})$ such that for any t_1, $t_2 \in (0,1)$ and $N \in \mathbb{N}$,*

$$\left| \int_{\mathbb{R}} e^{i((t_1-t_2)|\xi|^a - x\xi)} (1+\xi^2)^{-\frac{\alpha}{2}} e^{-(t_1^\gamma + t_2^\gamma)|\xi|^a} \mu\left(\frac{\xi}{N}\right) d\xi \right| \leqslant K(x)$$

for all $x \in \mathbb{R}$.

The assumption here that $\alpha < \frac{1}{2}$ for $\gamma < \frac{a}{a-1}$ is purely for technical reasons and since only minimal choices of α are of interest in proving Theorem 6.1.2, it will have no impact on the usefulness of this lemma.

The sufficiency of this lemma in establishing the desired boundedness follows from an application of the Kolmogorov–Seliverstov–Plessner method, similarly to Carleson's original work in [46]. Indeed, assuming this lemma to be true, fix a positive, even $\eta \in \mathscr{S}(\mathbb{R})$ supported in $[-1,1]$ and equal to 1 in $[-\frac{1}{2},\frac{1}{2}]$ and radially decreasing. Also, fix a measurable function $t:\mathbb{R}\to(0,1)$ and define for each $N\in\mathbb{N}$,

$$P^{t(x)}_{a,\gamma,N} f(x) := \eta\left(\frac{x}{N}\right)\int_{\mathbb{R}} \widehat{f}(\xi) e^{it(x)|\xi|^a} e^{-t^{\gamma}(x)|\xi|^a} e^{ix\xi}\eta\left(\frac{\xi}{N}\right) d\xi.$$

To establish that $\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ for all $s > \frac{1}{4}a\left(1-\frac{1}{\gamma}\right)$, it suffices to prove that

$$\|P^{t(\cdot)}_{a,\gamma,N} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for any $N\in\mathbb{N}$ with constant independent of N and t. This method of linearising the maximal operator, $P^*_{a,\gamma}$, by introducing the function $t(x)$ is exactly the same as the method used in Part I to linearise the Carleson maximal operator. The freedom of choice in t and the independence of the bound from this choice, implies boundedness of the maximal operator since t may be chosen (depending on f) to choose values of t where the supremum in the maximal operator is "essentially attained" (that is, attained up to a fixed constant multiple).

By duality, this bound is equivalent to showing that for any $g\in L^2(\mathbb{R})$ with $\|g\|_{L^2(\mathbb{R})} = 1$,

$$\left|\int_{\mathbb{R}} (P^{t(x)}_{a,\gamma,N} f)(x)\overline{g(x)}\, dx\right| \lesssim \|f\|_{H^s(\mathbb{R})}.$$

However, by Fubini's theorem and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} (P_{a,\gamma,N}^{t(x)} f)(x)\overline{g(x)}\,dx\right| \\
=\ &\left|\int_{\mathbb{R}} \widehat{f}(\xi)(1+\xi^2)^{\frac{s}{2}}(1+\xi^2)^{-\frac{s}{2}}\eta\left(\frac{\xi}{N}\right)\int_{\mathbb{R}} e^{it(x)|\xi|^a}e^{-t^\gamma(x)|\xi|^a}e^{ix\xi}\overline{g(x)}\eta\left(\frac{x}{N}\right)\,dx\,d\xi\right| \\
\leqslant\ &\|f\|_{H^s(\mathbb{R})}\left(\int_{\mathbb{R}}(1+\xi^2)^{-s}\eta^2\left(\frac{\xi}{N}\right)\left|\int_{\mathbb{R}} e^{it(x)|\xi|^a}e^{-t^\gamma(x)|\xi|^a}e^{ix\xi}\overline{g(x)}\,dx\right|^2 d\xi\right)^{\frac{1}{2}} \\
=\ &\|f\|_{H^s(\mathbb{R})}\left|\int_{\mathbb{R}}(1+\xi^2)^{-s}\eta^2\left(\frac{\xi}{N}\right)\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t(x)-t(y))|\xi|^a}e^{-(t^\gamma(x)+t^\gamma(y))|\xi|^a}e^{i(x-y)\xi}\overline{g(x)}g(y)\right. \\
&\left.\times\,\eta\left(\frac{x}{N}\right)\eta\left(\frac{y}{N}\right)\,dx\,dy\,d\xi\right|^{\frac{1}{2}} \\
\lesssim\ &\|f\|_{H^s(\mathbb{R})}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|g(x)||g(y)|\left|\int_{\mathbb{R}} e^{i((t(x)-t(y))|\xi|^a-(y-x)\xi)}(1+\xi^2)^{-s}e^{-(t^\gamma(x)+t^\gamma(y))|\xi|^a}\right.\right. \\
&\left.\left.\times\,\eta^2\left(\frac{\xi}{N}\right)d\xi\right|\,dx\,dy\right)^{\frac{1}{2}}.
\end{aligned}
$$

By Lemma 6.2.1 (where $\alpha = 2s$, $\mu = \eta^2$, $t_1 = t(x)$ and $t_2 = t(y)$) and a further application of the Cauchy–Schwarz inequality, there exists a function $K \in L^1(\mathbb{R})$, independent of t and N, such that this quantity can be bounded by

$$\|f\|_{H^s(\mathbb{R})}\||K|*|g|\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R})}^{\frac{1}{2}},$$

which is bounded by $\|f\|_{H^s(\mathbb{R})}\|K\|_{L^1(\mathbb{R})}^{\frac{1}{2}}$ by Young's convolution inequality (see, for example, [66, Thm. 1.2.12, p. 21]) and the fact that $\|g\|_{L^2(\mathbb{R})} = 1$. This establishes the desired boundedness of $P_{a,\gamma}^*$.

Lemma 6.2.1 is based on Lemmata 2.1, 2.2, 2.3 and 2.4 from [133] and its proof given here follows a similar strategy to the proofs of those lemmata. Note that of these four lemmata, Lemma 2.1 was proved in [131] (where it is cited as having been originally proved implicitly in [70] using a method from [129]) and Lemma 2.2 was proved in [132].

With the sufficiency of Lemma 6.2.1 in proving the positive part of Theorem 6.1.2 established, the remainder of this section will be devoted to its proof.

To begin with, for each $\varepsilon > 0$, define the function $h_\varepsilon(\xi) := e^{-\varepsilon|\xi|^a}$. It is claimed that for $\xi \neq 0$,

$$|h'_\varepsilon(\xi)| \lesssim \frac{1}{|\xi|} \quad \text{and} \quad |h''_\varepsilon(\xi)| \lesssim \frac{1}{|\xi|^2}$$

with constant independent of ε. Indeed, note that $h'_\varepsilon(\xi) = -\operatorname{sgn}(\xi)\varepsilon a|\xi|^{a-1}e^{-\varepsilon|\xi|^a}$, so

$$|h'_\varepsilon(\xi)| \leqslant \frac{a}{|\xi|} \max_{y\in\mathbb{R}^+} y e^{-y} \lesssim \frac{1}{|\xi|}.$$

Similarly,

$$|h''_\varepsilon(\xi)| \lesssim \frac{1}{|\xi|^2} \max_{y\in\mathbb{R}^+} y e^{-y} + \frac{1}{|\xi|^2} \max_{y\in\mathbb{R}^+} y^2 e^{-y} \lesssim \frac{1}{|\xi|^2}.$$

In addition to these facts, to prove Lemma 6.2.1, the following version of Van der Corput's Lemma will be required:

Lemma (Van der Corput) *Let F, $\psi \in C^\infty([a,b])$ for some fixed a, $b \in \mathbb{R}$ with $a < b$ and F real-valued. Assume that for some $k \in \mathbb{N}$ and $\lambda > 0$, $|F^{(k)}(x)| \geqslant \lambda$ for all $x \in [a,b]$. If $k = 1$ then assume further that F' is monotonic. Then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right| \lesssim \lambda^{-\frac{1}{k}} \left(\sup_{x\in[a,b]} |\psi(x)| + \int_a^b |\psi'(x)|\,dx\right)$$

with constant independent of F, ψ, a and b.

This is one of a number of possible formulations of Van der Corput's Lemma, a proof of which may be found in [136, Ch. VIII], for example. It has the following corollary, which will be useful here:

Corollary 6.2.2 *Let F, $\psi \in C^\infty([a,b])$ for some fixed a, $b \in \mathbb{R}$ with $a < b$ and F real-valued. Assume that ψ' changes sign at most N times on $[a,b]$ for some $N \in \mathbb{N}$ and that for some $k \in \mathbb{N}$ and $\lambda > 0$, $|F^{(k)}(x)| \geqslant \lambda$ for all $x \in [a,b]$. If $k = 1$ then assume further that F' is monotonic. Then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right| \lesssim_N \lambda^{-\frac{1}{k}} \sup_{x\in[a,b]} |\psi(x)|$$

with constant independent of F, ψ, a and b.

Proof By Van der Corput's Lemma, it will suffice to show here that

$$\int_a^b |\psi'(x)|\,dx \lesssim_N \sup_{x\in[a,b]} |\psi(x)|.$$

By hypothesis, $[a,b]$ can be written as $\bigcup_{n=1}^{N} I_n$ where the I_n are sub-intervals of $[a,b]$ on which ψ' is either non-negative or non-positive. Consequently,

$$\int_a^b |\psi'(x)|\,dx = \sum_{n=1}^{N} \left| \int_{I_n} \psi'(x)\,dx \right|,$$

but by the Fundamental Theorem of Calculus, this quantity is bounded by

$$\sum_{n=1}^{N} \sup_{x\in I_n} |\psi(x)| \leqslant N \sup_{x\in[a,b]} |\psi(x)|$$

and so the result follows. □

To proceed with the proof of Lemma 6.2.1, assume without loss of generality that $t_2 < t_1$ and set $t := t_1 - t_2$ and $\varepsilon := t_1^\gamma + t_2^\gamma$. Also, define $F(\xi) = t|\xi|^a - x\xi$ and $G(\xi) = (1 + \xi^2)^{-\frac{\alpha}{2}} e^{-\varepsilon|\xi|^a} \mu(\frac{\xi}{N})$, so that the integral in Lemma 6.2.1 can be rewritten as

$$\int_{\mathbb{R}} e^{iF(\xi)} G(\xi)\,d\xi.$$

The letter ρ will be used to denote $(\frac{|x|}{ta})^{\frac{1}{a-1}}$, a (possibly) stationary point of F.

Fixing a large constant $C_0 \in \mathbb{R}^+$, the cases of $|x| \leqslant C_0$ and $|x| \geqslant C_0$ will be considered separately.

6.2.1 The Case of Small x

Split the integral from Lemma 6.2.1 as $A + B$ where

$$\begin{aligned} A &:= \int_{|\xi|\leqslant|x|^{-1}} e^{i(t|\xi|^a - x\xi)}(1+\xi^2)^{-\frac{\alpha}{2}} e^{-\varepsilon|\xi|^a} \mu\left(\frac{\xi}{N}\right) d\xi, \\ B &:= \int_{|\xi|\geqslant|x|^{-1}} e^{i(t|\xi|^a - x\xi)}(1+\xi^2)^{-\frac{\alpha}{2}} e^{-\varepsilon|\xi|^a} \mu\left(\frac{\xi}{N}\right) d\xi. \end{aligned}$$

The first integral, A, can be bounded trivially by simply observing that

$$|A| \lesssim \int_{|\xi|\leqslant|x|^{-1}} (1+\xi^2)^{-\frac{\alpha}{2}} d\xi \lesssim 1 + |x|^{\alpha-1}.$$

As $\min(0, \alpha-1) > -1$, this establishes a suitable estimate on A, showing that it is bounded by a function that is integrable for small x.

To estimate B, begin by considering the case where $|x|^a \leqslant \frac{t}{2}$. This ensures that $\rho \leqslant |x|^{-1}$ and thus that the phase of the integrand (the function F) is never stationary in the region of integration for B.

By symmetry, it will suffice to bound B with the range of integration restricted to positive values of ξ. By direct calculation, for such ξ, $F'(\xi) = at\xi^{a-1} - x$, so it can be seen that F' is monotonic. Further, given that $|\xi| \geqslant |x|^{-1}$ and $\frac{t}{|x|^{a-1}} \geqslant 2|x|$, it can be seen that

$$|F'(\xi)| \geqslant |x|(2a-1) > |x|.$$

Now, G is radially decreasing, so for $\xi > |x|^{-1}$, G' is non-positive. As such, by the corollary to Van der Corput's Lemma, it follows that

$$\left|\int_{\xi>|x|^{-1}} e^{iF(\xi)} G(\xi)\, d\xi\right| \lesssim \frac{1}{|x|} \sup_{\xi>|x|^{-1}} |G(\xi)|.$$

Trivially, $|G(\xi)| \lesssim (1+\xi^2)^{-\frac{\alpha}{2}} \lesssim |x|^\alpha$ for $\xi > |x|^{-1}$ and hence $B \lesssim |x|^{\alpha-1}$. Again, given that $\alpha - 1 > -1$, a suitable estimate for B holds in the case of $|x|^a \leqslant \frac{t}{2}$.

To complete the proof of the lemma in the case of $|x| \leqslant C_0$, it remains to bound B in the case that $|x|^a \geqslant \frac{t}{2}$. As before, it suffices to consider only positive ξ. To proceed, fix a small constant, δ, and a large constant, K. The range of integration will be split into the following regions:

$$\begin{aligned} I_1 &\coloneqq \{\xi \geqslant |x|^{-1} : \xi \leqslant \delta\rho\}, \\ I_2 &\coloneqq \{\xi \geqslant |x|^{-1} : \xi \in [\delta\rho, K\rho]\}, \\ I_3 &\coloneqq \{\xi \geqslant |x|^{-1} : \xi \geqslant K\rho\}, \end{aligned}$$

recalling that $\rho = (\frac{|x|}{ta})^{\frac{1}{a-1}}$. For each $j \in \{1,2,3\}$, the integral in B restricted to the region I_j will be denoted by J_j.

This splitting isolates a neighbourhood around the point of (possible) stationary phase of the integrand (I_2) from the remaining range of integration either side (I_1 and I_3). The latter regions will be bounded using a lower bound on F' and an application of Van der Corput's Lemma, as before. Indeed, for $\xi \in I_1$, it can be seen that $at\xi^{a-1} \leqslant \delta^{a-1}|x| \leqslant \frac{|x|}{2}$, hence $|F'(\xi)| = |at\xi^{a-1} - x| \geqslant \frac{|x|}{2}$. Similarly, for $\xi \in I_3$, it can be seen that $at\xi^{a-1} \geqslant K^{a-1}|x| \geqslant 2|x|$, so $|F'(\xi)| \geqslant \frac{|x|}{2}$. From before,

$$\sup_{\xi > |x|^{-1}} |G(\xi)| \lesssim |x|^\alpha,$$

so by the corollary to Van der Corput's Lemma,

$$|J_1|,\ |J_3| \lesssim |x|^{-1}|x|^\alpha = |x|^{\alpha-1},$$

which completes the estimates on J_1 and J_3 as before.

Unsurprisingly, the estimate on J_2 is more delicate, although it is still attained using Van der Corput's Lemma, this time with the second derivative of F bounded below. To begin with, assume that $\gamma \geqslant \frac{a}{a-1}$. For any ξ in I_2, it is the case that $\xi \sim \rho$. Given that $F''(\xi) = a(a-1)t\xi^{a-2}$, it follows that $|F''(\xi)| \gtrsim t^{\frac{1}{a-1}}|x|^{\frac{a-2}{a-1}}$. As before, but now using that $\xi \gtrsim \rho$ instead of simply that $\xi \geqslant |x|^{-1}$, it is also the case that $\sup_{\xi\in I_2}|G(\xi)| \lesssim \rho^{-\alpha}$. Consequently, by the corollary to Van der Corput's Lemma with $k=2$,

$$|J_2| \lesssim t^{-\frac{1}{2(a-1)}}|x|^{-\frac{(a-2)}{2(a-1)}}\rho^{-\alpha} \approx t^{\frac{1}{a-1}(\alpha-\frac{1}{2})}|x|^{\frac{1}{a-1}(1-\frac{1}{2}a-\alpha)}.$$

Since $\gamma \geqslant \frac{a}{a-1}$, it is necessarily the case that $\alpha - \frac{1}{2} > 0$. Using further the assumption that $|x|^a \geqslant \frac{t}{2}$, it follows that

$$t^{\frac{1}{a-1}(\alpha-\frac{1}{2})} \lesssim |x|^{\frac{a}{a-1}(\alpha-\frac{1}{2})},$$

so

$$|J_2| \lesssim |x|^{\frac{a}{a-1}(\alpha-\frac{1}{2})}|x|^{\frac{1}{a-1}(1-\frac{1}{2}a-\alpha)} = |x|^{\alpha-1},$$

which completes the desired estimate.

It remains only to consider the case of $\gamma < \frac{a}{a-1}$. For small x, this is the most difficult part of the argument, requiring the full strength of the hypothesis that $\alpha > \frac{1}{2}a(1-\frac{1}{\gamma})$ for the first time. The lower bound, $|F''(\xi)| \geqslant t^{\frac{1}{a-1}}|x|^{\frac{a-2}{a-1}}$ will be used again in another application of Van der Corput's Lemma, but instead of using the fact that $|x|^a \geqslant \frac{t}{2}$, improved estimates on G will be required, depending on the exponential decay factor in G (which corresponds to the decay introduced to the multiplier for the maximal operator by the imaginary part of the time variable). Indeed, note that

$$\sup_{\xi\in I_2}|G(\xi)| \lesssim \rho^{-\alpha}e^{-\delta^a\varepsilon\rho^a}.$$

As such, by the corollary to Van der Corput's Lemma with $k = 2$,

$$\begin{aligned} |J_2| &\lesssim t^{-\frac{1}{2(a-1)}} |x|^{-\frac{a-2}{2(a-1)}} \rho^{-\alpha} e^{-\delta^a \varepsilon \rho^a} \\ &\approx t^{\frac{1}{a-1}(\alpha-\frac{1}{2})} |x|^{\frac{1}{a-1}(-\alpha-\frac{1}{2}(a-2))} e^{-\delta^a (t_1^\gamma + t_2^\gamma)|x|^{\frac{a}{a-1}} t^{-\frac{a}{a-1}}}. \end{aligned}$$

Further, observing that $t_1^\gamma + t_2^\gamma \geqslant 2^{-\gamma}(t_1 + t_2)^\gamma \gtrsim t^\gamma$ (this is a manifestation of the equivalence of finite-dimensional norms), it can be seen that there exists a small constant $c_0 > 0$ such that

$$|J_2| \lesssim t^{\frac{1}{a-1}(\alpha-\frac{1}{2})} |x|^{\frac{1}{a-1}(-\alpha-\frac{1}{2}(a-2))} e^{-\delta^a c_0 t^{\gamma-\frac{a}{a-1}} |x|^{\frac{a}{a-1}}}.$$

Noting that for any $y, \beta > 0$, the inequality

$$e^{-y} \lesssim_\beta y^{-\beta}$$

holds, it follows that for any $\beta > 0$,

$$\begin{aligned} |J_2| &\lesssim t^{\frac{1}{a-1}(\alpha-\frac{1}{2})} |x|^{\frac{1}{a-1}(-\alpha-\frac{1}{2}(a-2))} t^{-\beta(\gamma-\frac{a}{a-1})} |x|^{-\frac{\beta a}{a-1}} \\ &= \frac{t^{\frac{1}{a-1}(\alpha-\frac{1}{2})}}{t^{\beta(\gamma-\frac{a}{a-1})}} \frac{1}{|x|^{\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a)}}. \end{aligned}$$

Choose β such that $\frac{1}{a-1}(\alpha - \frac{1}{2}) = \beta(\gamma - \frac{a}{a-1})$, that is $\beta = \frac{\alpha-\frac{1}{2}}{(a-1)\gamma-a}$, noting that β is genuinely positive since $\gamma < \frac{a}{a-1}$ and $\alpha < \frac{1}{2}$. It follows that $|J_2| \lesssim \frac{1}{|x|^k}$ where

$$k = \frac{1}{a-1}\left(\alpha + \frac{1}{2}(a-2) + \frac{a(\alpha-\frac{1}{2})}{(a-1)\gamma - a}\right),$$

so it remains only to show that $k < 1$ to establish integrability of this bound for small x. However,

$$k = \frac{1}{a-1}\left(\alpha\left(\frac{(a-1)\gamma}{(a-1)\gamma-a}\right) + \frac{1}{2}(a-2) - \frac{\frac{1}{2}a}{(a-1)\gamma-a}\right),$$

but $\gamma < \frac{a}{a-1}$, so $\frac{(a-1)\gamma}{(a-1)\gamma-a} < 0$, hence the fact that $\alpha > \frac{1}{2}a(1-\frac{1}{\gamma})$ implies that

$$k < \frac{1}{a-1}\left(\frac{1}{2}a\left(1-\frac{1}{\gamma}\right)\left(\frac{(a-1)\gamma}{(a-1)\gamma-a}\right)+\frac{1}{2}(a-2)-\frac{\frac{1}{2}a}{(a-1)\gamma-a}\right)=1.$$

The estimate for $|x| < C_0$ is thus established.

6.2.2 The Case of Large x

Here again the integral will be split into four regions, this time smoothly partitioned. To this end, define $\phi_0 \in \mathscr{S}(\mathbb{R})$ to be supported in $[-1,1]$ and equal to 1 in $[-\frac{1}{2},\frac{1}{2}]$ and $\phi_2 \in \mathscr{S}(\mathbb{R})$ to be supported in $[\delta\rho, K\rho]$ and equal to 1 in $[2\delta\rho, \frac{1}{2}K\rho]$, where, as before, δ is a small constant, K is a large constant and $\rho = (\frac{|x|}{ta})^{\frac{1}{a-1}}$. For the sake of simplicity, it is assumed that C_0 and δ have been chosen so that $\delta(\frac{|x|}{a})^{\frac{1}{a-1}} > 1$ (and hence $\delta\rho > 1$ and so the supports of ϕ_0 and ϕ_2 do not overlap). Define $\phi_1 := (1-\phi_2-\phi_0)\chi_{[\frac{1}{2},2\delta\rho]}$ and $\phi_3 := (1-\phi_2)\chi_{[\frac{1}{2}K\rho,\infty)}$. It is then the case that the ϕ_i form a partition of unity on the positive real line, that is to say that $(\phi_0+\phi_1+\phi_2+\phi_3)\chi_{[0,\infty)} \equiv \chi_{[0,\infty)}$.

Define $G_j := G\phi_j$ and let I_j represent the support of G_j for each $j \in \{0,1,2,3\}$, so that

$$\begin{aligned}
I_0 &:= [-1,1],\\
I_1 &:= [\tfrac{1}{2}, 2\delta\rho],\\
I_2 &:= [\delta\rho, K\rho],\\
I_3 &:= [\tfrac{1}{2}K\rho, \infty).
\end{aligned}$$

As before, this splitting isolates a region, I_2, around a point of possible stationary phase of the integrand from regions either side, I_1 and I_3. Localising the integrand to the region I_0 produces a term that has a similar role to integral A from the previous section.

By symmetry it suffices to estimate

$$J_j \coloneqq \int_{I_j} e^{iF} G_j$$

for each $j \in \{0,1,2,3\}$ to conclude the full estimate on $\displaystyle\int_{\mathbb{R}} e^{iF(\xi)} G(\xi)\, d\xi$.

In the case of J_0, writing $e^{iF}G$ as $(e^{-ix\xi})(e^{it|\xi|^a} G_0(\xi))$ and integrating by parts twice yields that

$$|J_0| \leqslant \frac{1}{x^2} \int_{-1}^{1} \left| \frac{d^2}{d\xi^2} (e^{it|\xi|^a} G_0(\xi)) \right| d\xi.$$

By direct calculation and the triangle inequality,

$$\left| \frac{d^2}{d\xi^2} (e^{it|\xi|^a} G_0(\xi)) \right| \lesssim |\xi|^{a-2} |G_0(\xi)| + |\xi|^{2a-2} |G_0(\xi)| + |\xi|^{a-1} |G_0'(\xi)| + |G_0''(\xi)|.$$

Now, $G_0(\xi) = (1+\xi^2)^{-\frac{\alpha}{2}} e^{-\varepsilon|\xi|^a} \mu(\frac{\xi}{N}) \phi_0(\xi)$, so for $\xi \in [-1,1]$, it is clear that $|G_0(\xi)| \lesssim 1$. Further, the first derivatives of all terms in the product defining G_0 are bounded for $\xi \in [-1,1]$, so $|G_0'(\xi)| \lesssim 1$ also. The second derivatives of all terms in the product defining G_0 are also bounded for $\xi \in [-1,1]$ with the possible exception of $\frac{d^2}{d\xi^2} e^{-\varepsilon|\xi|^a}$. However, by the triangle inequality,

$$\left| \frac{d^2}{d\xi^2} e^{-\varepsilon|\xi|^a} \right| \leqslant \varepsilon a(a-1) |\xi|^{a-2} e^{-\varepsilon|\xi|^a} + (a\varepsilon|\xi|^{a-1})^2 e^{-\varepsilon|\xi|^a} \lesssim |\xi|^{a-2} + 1.$$

Given that $a > 1$, this expression is integrable on $[-1,1]$, and it hence follows that

$$\int_{-1}^{1} \left| \frac{d^2}{d\xi^2} (e^{it|\xi|^a} G_0(\xi)) \right| d\xi \lesssim 1,$$

so $|J_0| \lesssim x^{-2}$, which establishes that J_0 is bounded by an integrable function for large x, as desired.

For $j \in \{1,3\}$, integrating by parts twice yields that

$$\begin{aligned} |J_j| &= \Big|\int_{I_j} e^{iF(\xi)} \left(-\frac{{G_j}''(\xi)}{(F'(\xi))^2} + \frac{3{G_j}'(\xi)F''(\xi)}{(F'(\xi))^3} + \frac{G_j(\xi)F'''(\xi)}{(F'(\xi))^3} - \frac{3G_j(\xi)(F''(\xi))^2}{(F'(\xi))^4} \right) d\xi \Big| \\ &\lesssim \int_{I_j} \frac{1}{(F'(\xi))^2} \left(|{G_j}''(\xi)| + \frac{|F''(\xi)|}{|F'(\xi)|}|{G_j}'(\xi)| + \frac{|F'''(\xi)|}{|F'(\xi)|}|G_j(\xi)| + \frac{|F''(\xi)|^2}{|F'(\xi)|^2}|G_j(\xi)| \right) d\xi. \end{aligned}$$

Given that $\xi > 0$, by direct calculation,

$$F(\xi) = t\xi^a - x\xi, \quad F'(\xi) = at\xi^{a-1} - x,$$
$$F''(\xi) = a(a-1)t\xi^{a-2}, \quad F'''(\xi) = a(a-1)(a-2)t\xi^{a-3}.$$

For $\xi \in I_1$, since $\xi \leqslant 2\delta\rho$, it follows that $at\xi^{a-1} \leqslant at2^{a-1}\delta^{a-1}\rho^{a-1} = 2^{a-1}\delta^{a-1}|x|$. Consequently, $|F'(\xi)| \gtrsim |x|$ and hence it is also true that $|F'(\xi)| \gtrsim at\xi^{a-1}$. Similarly, for $\xi \in I_3$, since $\xi \geqslant \frac{1}{2}K\rho$, it follows that $at\xi^{a-1} \geqslant at2^{1-a}K^{a-1}\rho^{a-1} = 2^{1-a}K^{a-1}|x|$, hence $|F'(\xi)| \gtrsim |x|$ and $|F'(\xi)| \gtrsim at\xi^{a-1}$ as well. In either case,

$$\frac{|F''(\xi)|}{|F'(\xi)|} \lesssim \xi^{-1} \quad \text{and} \quad \frac{|F'''(\xi)|}{|F'(\xi)|} \lesssim \xi^{-2}.$$

It follows that for $j \in \{1,3\}$,

$$|J_j| \lesssim \frac{1}{|x|^2} \int_{I_j} (|{G_j}''(\xi)| + |\xi|^{-1}|{G_j}'(\xi)| + 2|\xi|^{-2}|G_j(\xi)|)\, d\xi$$

Recalling that $h_\varepsilon(\xi) := e^{-\varepsilon|\xi|^a}$ satisfies the estimates $|h'_\varepsilon(\xi)| \lesssim \frac{1}{|\xi|}$ and $|h''_\varepsilon(\xi)| \lesssim \frac{1}{|\xi|^2}$, with constants independent of ε, and that

$$G_j(\xi) := (1+\xi^2)^{-\frac{\alpha}{2}} e^{-\varepsilon|\xi|^a} \mu\left(\frac{\xi}{N}\right) \phi_j(\xi),$$

it is easily seen that for any $j \in \{1,2,3\}$,

$$|G_j(\xi)| \lesssim \frac{1}{|\xi|^\alpha}, \quad |{G_j}'(\xi)| \lesssim \frac{1}{|\xi|^{\alpha+1}}, \quad |{G_j}''(\xi)| \lesssim \frac{1}{|\xi|^{\alpha+2}}.$$

Consequently, for $j \in \{1,3\}$,

$$|J_j| \lesssim \frac{1}{|x|^2} \int_{I_j} \frac{1}{|\xi|^{\alpha+2}}\, d\xi \lesssim \frac{1}{|x|^2},$$

which provides a suitable bound on J_1 and J_3.

To bound J_2, following the same steps as in the bound for J_2 when $|x| \leqslant C_0$, it can be seen that for any $\xi \in I_2$,

$$|F''(\xi)| \gtrsim t^{\frac{1}{a-1}} |x|^{\frac{a-2}{a-1}}$$

and that

$$\sup_{\xi \in I_2} |G_2(\xi)| \lesssim \rho^{-\alpha} e^{-\delta^a \varepsilon \rho^a},$$

so given a suitable choice of ϕ_2, by the corollary to Van der Corput's Lemma,

$$|J_2| \lesssim t^{\frac{1}{a-1}(\alpha-\frac{1}{2})} |x|^{\frac{1}{a-1}(-\alpha-\frac{1}{2}(a-2))} e^{-\delta^a c_0 t^{\gamma-\frac{a}{a-1}} |x|^{\frac{a}{a-1}}}$$

for some small constant $c_0 > 0$. If $\gamma = \frac{a}{a-1}$, then noting that $\alpha > \frac{1}{2}$ and that $\alpha + \frac{1}{2}(a-2) > 0$, it follows that $|J_2| \lesssim e^{-\delta^a c_0 |x|^{\frac{a}{a-1}}}$ and the estimate is complete. Otherwise, proceeding as before, using the fact that for any y, $\beta > 0$,

$$e^{-y} \lesssim_\beta y^{-\beta},$$

it follows that for any $\beta > 0$,

$$|J_2| \lesssim \frac{t^{\frac{1}{a-1}(\alpha-\frac{1}{2})}}{t^{\beta(\gamma-\frac{a}{a-1})}} \frac{1}{|x|^{\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a)}}.$$

If $\gamma < \frac{a}{a-1}$, rewrite this estimate as

$$|J_2| \lesssim \frac{t^{\beta(\frac{a}{a-1}-\gamma)}}{t^{\frac{1}{a-1}(\frac{1}{2}-\alpha)}} \frac{1}{|x|^{\frac{1}{a-1}(\alpha+\frac{1}{2}(a-2)+\beta a)}}$$

and note that β can be set as large as is desired to conclude a suitable estimate.

If $\gamma > \frac{a}{a-1}$, choose $\beta = \frac{\alpha-\frac{1}{2}}{(a-1)\gamma-a}$ as in the case of $|x| \leqslant C_0$, noting that it is still the case that this choice of β is positive, as $\alpha > \frac{1}{2}$ and $\gamma > \frac{a}{a-1}$. As before, it can thus be concluded that $|J_2| \lesssim \frac{1}{|x|^k}$ where $k = \frac{1}{a-1}\left(\alpha\left(\frac{(a-1)\gamma}{(a-1)\gamma-a}\right) + \frac{1}{2}(a-2) - \frac{\frac{1}{2}a}{(a-1)\gamma-a}\right)$ and it remains to show that $k > 1$ in this case. However, since $\gamma > \frac{a}{a-1}$, it is necessarily the case that $\frac{(a-1)\gamma}{(a-1)\gamma-a} > 0$, hence the fact that $\alpha > \frac{1}{2}a(1-\frac{1}{\gamma})$ implies that

$$k > \frac{1}{a-1}\left(\frac{1}{2}a\left(1-\frac{1}{\gamma}\right)\left(\frac{(a-1)\gamma}{(a-1)\gamma-a}\right) + \frac{1}{2}(a-2) - \frac{\frac{1}{2}a}{(a-1)\gamma-a}\right) = 1,$$

which completes the estimate on J_2 and the proof of Lemma 6.2.1.

6.3 Failure of Boundedness for Regularity Below the Critical Index

To complete the proof of Theorem 6.1.2, it remains to show that for $\gamma > 1$, the estimate $\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ cannot hold for $s < \frac{1}{4}a(1-\frac{1}{\gamma})$. In [132], Sjölin proved this for $a = \gamma = 2$, generalising the counterexample of Dahlberg and Kenig from [51]. The proof given here is a further generalisation of this counterexample. Construction of counterexamples of this form will be discussed in Section 7.5 within the context of some related work.

Fix $\gamma > 1$ and $s < \frac{1}{4}a(1-\frac{1}{\gamma})$ and for each $v \in (0, v_0)$ for some small $v_0 > 0$, choose $g_v \in \mathscr{S}(\mathbb{R})$ to be a positive, even, real-valued function, supported in $[-v^{(a-1)-\frac{a}{\gamma}}, v^{(a-1)-\frac{a}{\gamma}}]$ and such that $g_v(\xi) = 1$ for $\xi \in [-\frac{1}{2}v^{(a-1)-\frac{a}{\gamma}}, \frac{1}{2}v^{(a-1)-\frac{a}{\gamma}}]$. Define the function f_v such that $\widehat{f_v}(\xi) = v g_v(v\xi + \frac{1}{v})$ and note that

$$\begin{aligned} \|f_v\|^2_{\dot{H}^s(\mathbb{R})} &= v^2 \int_{\mathbb{R}} \left|g_v\left(v\xi + \frac{1}{v}\right)\right|^2 |\xi|^{2s}\, d\xi \\ &= \frac{v^2}{v^{1+2s}} \int_{\mathbb{R}} \left|g_v\left(\xi + \frac{1}{v}\right)\right|^2 |\xi|^{2s}\, d\xi. \end{aligned}$$

For the integrand above to be non-zero, given the support of g_v, it is necessarily the case that $\xi + \frac{1}{v} \in [-v^{(a-1)-\frac{a}{\gamma}}, v^{(a-1)-\frac{a}{\gamma}}]$, hence $|\xi| \leqslant v^{(a-1)-\frac{a}{\gamma}} + \frac{1}{v}$. Since $v \in (0,1)$ and $(a-1) - \frac{a}{\gamma} > -1$, given that $\gamma > 1$, it follows that $|\xi| \lesssim \frac{1}{v}$, so

$$\|f_v\|^2_{\dot{H}^s(\mathbb{R})} \lesssim v^{1-2s} v^{(a-1)-\frac{a}{\gamma}} v^{-2s} = v^{a-4s-\frac{a}{\gamma}}.$$

Since $s < \frac{1}{4}a(1-\frac{1}{\gamma})$, it can be concluded that $\|f_v\|_{\dot{H}^s(\mathbb{R})} \to 0$ as $v \to 0$. Now, since $\|f\|_{H^s(\mathbb{R})} \sim \|f\|_{L^2(\mathbb{R})} + \|f\|_{\dot{H}^s(\mathbb{R})}$ and given that the above argument also provides that $\|f_v\|_{L^2(\mathbb{R})} \to 0$ as $v \to 0$, it follows that $\|f_v\|_{H^s(\mathbb{R})} \to 0$ as $v \to 0$. It thus now suffices to show that there exists a choice of t, depending on x and v, such that the $L^2(\mathbb{R})$ norm in x of $P^t_{a,\gamma} f_v(x)$ is bounded below, uniformly in v, since $P^*_{a,\gamma} f_v(x)$ is trivially pointwise bounded below by $P^t_{a,\gamma} f_v(x)$ for any particular choices of t.

Note first that

$$P^t_{a,\gamma} f_v(x) = \int_{\mathbb{R}} e^{i(x\xi + t|\xi|^a)} e^{-t^\gamma |\xi|^a} v g_v\left(v\xi + \frac{1}{v}\right) d\xi.$$

Substituting $\eta = v\xi + \frac{1}{v}$ and removing a unimodular term that does not depend on η from the integrand,

$$|P^t_{a,\gamma} f_v(x)| = \left| \int_{\mathbb{R}} e^{i(x\frac{\eta}{v} + t|\frac{\eta}{v} - \frac{1}{v^2}|^a)} e^{-t^\gamma |\frac{\eta}{v} - \frac{1}{v^2}|^a} g_v(\eta)\, d\eta \right|.$$

Define

$$\begin{aligned} F_{x,t,v}(\eta) &= x\frac{\eta}{v} + t\left|\frac{\eta}{v} - \frac{1}{v^2}\right|^a - \frac{t}{v^{2a}}, \\ G_{t,v}(\eta) &= t^\gamma \left|\frac{\eta}{v} - \frac{1}{v^2}\right|^a. \end{aligned}$$

Observing that $e^{-i\frac{t}{v^{2a}}}$ is unimodular and does not depend on η, it is clear that $e^{i(x\frac{\eta}{v}+t|\frac{\eta}{v}-\frac{1}{v^2}|^a)}$ in the above integral can be replaced with $e^{iF_{x,t,v}(\eta)}$. Now, bounding $|P^t_{a,\gamma}f_v(x)|$ below by $|\mathrm{Re}(P^t_{a,\gamma}f_v(x))|$ and recalling that g_v is supported in $[-v^{(a-1)-\frac{a}{\gamma}}, v^{(a-1)-\frac{a}{\gamma}}]$, it follows that

$$|(P^t_{a,\gamma}f_v)(x)| \geqslant \left| \int_{-v^{(a-1)-\frac{a}{\gamma}}}^{v^{(a-1)-\frac{a}{\gamma}}} \cos(F_{x,t,v}(\eta))e^{-G_{t,v}(\eta)} g_v(\eta)\, d\eta \right|.$$

By the binomial expansion, for $|\eta| \leqslant v^{(a-1)-\frac{a}{\gamma}}$,

$$\begin{aligned} \left|\frac{\eta}{v} - \frac{1}{v^2}\right|^a &= \left(\frac{1}{v^2} - \frac{\eta}{v}\right)^a \\ &= \frac{1}{v^{2a}} - \frac{a\eta}{v^{2(a-1)+1}} + O\left(\frac{\eta^2}{v^{2(a-2)+2}}\right), \end{aligned}$$

since $(a-1)-\frac{a}{\gamma} > -1$. It follows that

$$F_{x,t,v}(\eta) = x\frac{\eta}{v} - ta\frac{\eta}{v^{2a-1}} + O\left(\frac{t\eta^2}{v^{2(a-1)}}\right).$$

Fix $x \in [0, v^{\frac{2a}{\gamma}-2(a-1)}]$ and choose t, depending on x, so as to eliminate the first two terms in the above expression for F, that is to say choose $t = \frac{xv^{2(a-1)}}{a}$ (which is contained in $(0,1)$, given the restriction on x, and is thus a valid choice for t). Then

$$F_{x,t,v}(\eta) = O(x\eta^2)$$

and hence

$$F_{x,t,v}(\eta) \lesssim v^{\frac{2a}{\gamma}-2(a-1)+2((a-1)-\frac{a}{\gamma})} = 1.$$

For sufficiently small v_0, the implicit constant here may be set to 1.

Similarly, observing that $|\frac{\eta}{v} - \frac{1}{v^2}|^a = O(\frac{1}{v^{2a}})$ by using only the leading term of the binomial expansion here, it follows that

$$\begin{aligned} G_{t,v}(\eta) &= x^\gamma v^{2\gamma(a-1)} a^{-\gamma} O\left(\frac{1}{v^{2a}}\right) \\ &= O(x^\gamma v^{2a\gamma-2\gamma-2a}), \end{aligned}$$

hence given that x was fixed to be in $[0, v^{\frac{2a}{\gamma}-2(a-1)}]$,

$$G_{t,v}(\eta) \lesssim v^{2a-2\gamma(a-1)+2a\gamma-2\gamma-2a} = 1.$$

Given these estimates, it is clear that $\cos(F_{x,t,v}(\eta))$ and $e^{-G_{t,v}(\eta)}$ can be bounded below by constants for $|\eta| \leqslant v^{(a-1)-\frac{a}{\gamma}}$ and hence $|P^t_{a,\gamma} f_v(x)| \gtrsim v^{(a-1)-\frac{a}{\gamma}}$ for $x \in [0, v^{\frac{2a}{\gamma}-2(a-1)}]$. As such,

$$\|P^t_{a,\gamma} f_v\|^2_{L^2(\mathbb{R})} \gtrsim v^{\frac{2a}{\gamma}-2(a-1)}(v^{(a-1)-\frac{a}{\gamma}})^2 = 1.$$

This establishes that $\|P^t_{a,\gamma} f_v\|_{L^2(\mathbb{R})}$, and hence also $\|P^*_{a,\gamma} f_v\|_{L^2(\mathbb{R})}$, is bounded below uniformly in v which completes the proof of the negative part of Theorem 6.1.2.

6.4 Further Remarks and the Proof of Theorem 6.1.3

Whilst Theorem 6.1.2 determines precisely the infimum of the values of $s \geqslant 0$ for which the estimate $\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ holds, it is remarked that for $a > 1$ and $\gamma > 0$, the question of whether boundedness holds at the critical exponent, $s = \frac{1}{4}a(1-\frac{1}{\gamma})^+$, remains unaddressed. The analogous problem of boundedness at the endpoint is still open in the case of the maximal operators with real-valued time, S^*_a, that is to say that the veracity of the bound $\|S^*_a f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^{\frac{a}{4}}(\mathbb{R})}$ is still unknown, even in the case of $a = 2$ where S^*_a is the Schrödinger maximal operator. Nonetheless, the following partial resolution can be established here:

Theorem 6.4.1 (Endpoint Estimates)

(i) For $a > 1$ and $\gamma \in (0,1]$, the estimate $\|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}$ holds for all $f \in L^2(\mathbb{R})$.

(ii) For $a > 1$, the estimate $\|P^*_{a,\frac{a}{a-1}} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}$ holds for all $f \in H^{\frac{1}{4}}(\mathbb{R})$.

Proof To prove case (i), observe first that by Lemma 6.1.4, it suffices to prove only the case of $\gamma = 1$. To do this, define $K : \mathbb{R} \to \mathbb{R}$ such that $\widehat{K}(\xi) = e^{i|\xi|^a} e^{-|\xi|^a}$ and for each $t \in (0,1)$, define $K_t := t^{-1}K(t^{-1}\cdot)$, so that $\widehat{K_t} = \widehat{K}(t\cdot)$. Then

$$\begin{aligned} S_a^{t+it} f(x) &= \int_{\mathbb{R}} \widehat{f}(\xi)\Big(\int_{\mathbb{R}} K_{t^{\frac{1}{a}}}(y) e^{-2\pi i y\xi}\, dy\Big) e^{ix\xi}\, d\xi \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i(\frac{x}{2\pi} - y)\xi}\, d\xi\Big) K_{t^{\frac{1}{a}}}(y)\, dy \\ &= f * K_{t^{\frac{1}{a}}}(\tfrac{x}{2\pi}), \end{aligned}$$

so

$$\sup_{t\in(0,1)} |S_a^{t+it} f(x)| \leqslant \sup_{t\in(0,1)} |f * K_{t^{\frac{1}{a}}}(\tfrac{x}{2\pi})|.$$

Now,

$$K(x) = \int_{\mathbb{R}} e^{(i-1)|\xi|^a} e^{2\pi i x\xi}\, d\xi = \frac{1}{2}\int_0^\infty e^{(i-1)\xi^a} \cos(2\pi x\xi)\, d\xi.$$

Since

$$\Big|\int_0^\infty e^{(1-i)\xi^a} \cos(2\pi x\xi)\, d\xi\Big| \leqslant \int_0^\infty e^{-\xi^a}\, d\xi = \Gamma(\tfrac{1}{a} + 1) < \infty,$$

it is clear that $K \in L^\infty(\mathbb{R})$. Further, integrating by parts twice and using that for any $c > -1$ it is also true that $\displaystyle\int_0^\infty \xi^c e^{-\xi^a}\, d\xi = \tfrac{1}{a}\Gamma(\tfrac{c+1}{a}) < \infty$, it follows for any $x \neq 0$ that $|K(x)| \lesssim |x|^{-2}$.* It is thus the case that

$$\sup_{t\in(0,1)} |S_a^{t+it} f(x)| \lesssim Mf(\tfrac{x}{2\pi}),$$

where M is the Hardy–Littlewood maximal operator (see, for example, [66, Thm. 2.1.10, p. 82]). By boundedness of M on $L^2(\mathbb{R})$, the result follows.

Case (ii) follows immediately since the proof given in Section 6.2 adapts without difficulty in the case of $\gamma = \frac{a}{a-1}$ to include the endpoint $s = \frac{1}{4}a(1 - \frac{1}{\gamma}) = \frac{1}{4}$.† $\square$

*For $a \in \mathbb{N}\backslash\{1\}$ and for any $N \in \mathbb{N}$, by integrating by parts N times, this estimate can be improved to $|K(x)| \lesssim |x|^{-N}$.

†The significance of $\gamma = \frac{a}{a-1}$ is that it represents a point of transition in the proof between the values of γ where the constraint on α is required in Section 6.2.1 (proof for small values of x) and those where it is required in Section 6.2.2 (proof for large values of x).

As mentioned in Section 6.1, a natural extension of Theorem 6.1.2 is to consider the values of s for which a local norm bound on the maximal operator holds, that is to say

$$\|P^*_{a,\gamma} f\|_{L^2([-1,1])} \lesssim \|f\|_{H^s(\mathbb{R})}.$$

The answer to this question turns out to be a remarkably straightforward consequence of the proof of Theorem 6.1.2 and some other previous work of Sjölin. Denoting by $s_a^{\mathrm{loc}}(\gamma)$ the infimum of the values of $s > 0$ for which this estimate holds, the following analogue of Theorem 6.1.2 can be established:

Theorem 6.4.2 *For $\gamma \in (0,\infty)$ and $a > 1$, $s_a^{\mathrm{loc}}(\gamma) = \min\left(\frac{1}{4}a\left(1-\frac{1}{\gamma}\right)^+, \frac{1}{4}\right)$.*

Proof To begin with, observe that it is trivially true that $\|P^*_{a,\gamma} f\|_{L^2([-1,1])} \leqslant \|P^*_{a,\gamma} f\|_{L^2(\mathbb{R})}$, so the global bounds from Theorem 6.1.2 automatically imply some local bounds and it is thus necessarily the case that $s_a^{\mathrm{loc}}(\gamma) \leqslant s_a(\gamma) = \frac{1}{4}a(1-\frac{1}{\gamma})^+$. Additionally, note that the counterexample given in Section 6.3 is also a counterexample for the local estimate whenever the choices of x over which $P^*_{a,\gamma} f_v$ is pointwise bounded below can be contained within $[-1,1]$. Since x is chosen to be in $[0, v^{\frac{2a}{\gamma}-2(a-1)}]$ for some small parameter v, this happens precisely when $\frac{2a}{\gamma} - 2(a-1) \geqslant 0$, that is when $\gamma \leqslant \frac{a}{a-1}$. It follows that $s_a^{\mathrm{loc}}(\gamma) = \frac{1}{4}a(1-\frac{1}{\gamma})^+$ for $\gamma \in (0, \frac{a}{a-1}]$.

In the proof of Lemma 6.2.1, if $\gamma \geqslant \frac{a}{a-1}$ and x is small, the only requirement on α is that it is greater than or equal to $\frac{1}{2}$ (which is used in the estimate on J_2).* Consequently, for such γ, it must be the case that $s_a^{\mathrm{loc}}(\gamma) \leqslant \frac{1}{4}$. Further, the fact that $s_a^{\mathrm{loc}}(\gamma) \geqslant \frac{1}{4}$ for all $\gamma \geqslant \frac{a}{a-1}$ can be deduced from the counterexample of Section 6.3 in the case of $\gamma = \frac{a}{a-1}$. Indeed, the counterexample function f_v defined in the case of $\gamma = \frac{a}{a-1}$ can also be used as input to the operator $P^t_{a,\gamma}$ for higher values of γ since the function $G_{t,v}(\eta)$ from Section 6.3 is non-increasing in γ (given the requirement that t is local). The theorem is thus established. □

*The proof uses that $\alpha > \frac{1}{2}$ at this point (from the hypothesis of Lemma 6.2.1), but only $\alpha \geqslant \frac{1}{2}$ is required.

It is remarked that the upper bound of $\frac{1}{4}$ for $s_a^{\mathrm{loc}}(\gamma)$ here is perhaps unsurprising in light of the fact that in 1987, Sjölin proved in [129] that for all $a > 1$,

$$\|S_a^* f\|_{L^2([-1,1])} \lesssim \|f\|_{H^s(\mathbb{R})}$$

if and only if $s \geqslant \frac{1}{4}$.

In the case of this local problem, the question of boundedness at the endpoint can be resolved in more cases. Indeed, the following holds:

Theorem 6.4.3 (Local Endpoint Estimates)

(i) For $a > 1$ and $\gamma \in (0,1]$, the estimate $\|P_{a,\gamma}^* f\|_{L^2([-1,1])} \lesssim \|f\|_{L^2(\mathbb{R})}$ holds for all $f \in L^2(\mathbb{R})$.

(ii) For $a > 1$ and $\gamma \geqslant \frac{a}{a-1}$, the estimate $\|P_{a,\gamma}^* f\|_{L^2([-1,1])} \lesssim \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}$ holds for all $f \in H^{\frac{1}{4}}(\mathbb{R})$.

These results follow immediately from the global endpoint estimates (Theorem 6.4.1) and Lemma 6.1.4.

Given Theorem 6.4.2 together with these endpoint results, the positive pointwise convergence statements of Theorem 6.1.3 follow from the boundedness of the maximal operators that has been established. The negative statements are a consequence of the Nikishin maximal principle, as discussed in Section 5.2, using the fact that the counterexample in Section 6.3 can equally well be used to show failure of weak boundedness of $P_{a,\gamma}^*$ (that is boundedness from $H^s(\mathbb{R})$ into $L^{2,\infty}([-1,1])$).*

6.5 Proof of Lemma 6.1.4

Recall that Lemma 6.1.4 was stated as follows:

*Strictly speaking, failure of boundedness from $H^s(\mathbb{R})$ into $L^{2,\infty}(E)$ for all sets E of sufficiently large measure strictly less than 2 is required here. To prove this requires only a straightforward adaptation of the counterexamples considered.

Lemma 6.1.4 *Let g and h be continuous functions mapping $[0,1]$ to $[0,1]$ such that $g(t) \leqslant h(t)$ for all $t \in (0,1)$. Then for any $a > 1$,*

$$\| \sup_{t\in(0,1)} |S_a^{t+ih(t)} f| \|_{L^2(\mathbb{R})} \lesssim \| \sup_{t\in(0,1)} |S_a^{t+ig(t)} f| \|_{L^2(\mathbb{R})}$$

for any $f \in \mathscr{S}(\mathbb{R})$.

Proof As mentioned in Section 6.1, this proof is a mild generalisation of the proof of the analogous Lemma 1 from [132]. Observe that

$$\begin{aligned} S_a^{t+ih(t)} f(x) &= \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{-h(t)|\xi|^a} e^{ix\xi}\, d\xi \\ &= \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{-g(t)|\xi|^a} e^{-(h(t)-g(t))|\xi|^a} e^{ix\xi}\, d\xi. \end{aligned}$$

Define $K : \mathbb{R} \to \mathbb{R}$ such that $\widehat{K}(\xi) = e^{-|2\pi\xi|^a}$ and for each $t \in (0,1)$, define $K_t := t^{-1} K(t^{-1} \cdot)$, so that $\widehat{K_t} = \widehat{K}(t\cdot)$. Then

$$\begin{aligned} S_a^{t+ih(t)} f(x) &= \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{-g(t)|\xi|^a} \Big(\int_{\mathbb{R}} K_{(h(t)-g(t))^{\frac{1}{a}}}(y) e^{-2\pi i y \frac{\xi}{2\pi}}\, dy \Big) e^{ix\xi}\, d\xi \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{-g(t)|\xi|^a} e^{i(x-y)\xi}\, d\xi \Big) K_{(h(t)-g(t))^{\frac{1}{a}}}(y)\, dy \\ &= (S_a^{t+ig(t)} f) * K_{(h(t)-g(t))^{\frac{1}{a}}}(x), \end{aligned}$$

so

$$\sup_{t\in(0,1)} |S_a^{t+ih(t)} f(x)| = \sup_{t\in(0,1)} |(S_a^{t+ig(t)} f) * K_{(h(t)-g(t))^{\frac{1}{a}}}(x)| \leqslant \sup_{u\in(0,1)} |(\sup_{t\in(0,1)} |S_a^{t+ig(t)} f|) * K_u(x)|.$$

Now,

$$K(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-|\xi|^a} e^{ix\xi}\, d\xi = \frac{1}{\pi} \int_0^\infty e^{-\xi^a} \cos(x\xi)\, d\xi.$$

This kernel can be treated the same as the kernel in the proof of Theorem 6.4.1. Indeed, since $\int_0^\infty e^{-\xi^a}\, d\xi = \Gamma(\frac{1}{a}+1) < \infty$, it is clear that $K \in L^\infty(\mathbb{R})$. Further, integrating by parts twice and using that for any $c > -1$ it is also true that $\int_0^\infty \xi^c e^{-\xi^a}\, d\xi = \frac{1}{a}\Gamma(\frac{c+1}{a}) < \infty$, it follows for any $x \neq 0$ that $|K(x)| \lesssim |x|^{-2}$. It is thus the case that

$$\sup_{u\in(0,1)} |(\sup_{t\in(0,1)} |S_a^{t+ig(t)} f|) * K_u(x)| \lesssim M(\sup_{t\in(0,1)} |S_a^{t+ig(t)} f|)(x),$$

where M is the Hardy–Littlewood maximal operator. By boundedness of M on $L^2(\mathbb{R})$, the lemma follows. $\square$

Chapter 7

Maximal Operators Related to Oscillatory Kernels of Schrödinger Type

7.1 Introduction

In Chapter 6, results for maximal operators associated to multipliers of the form $e^{it|\cdot|^a}$ for $a > 1$ were established. The multiplier corresponding to the case of $a = 2$ is associated with the solution operator for the Schrödinger equation, yet by considering that in this case the multiplier is a Gaussian-type function, by calculating its inverse Fourier transform it is clear that the kernel associated to the solution operator for the Schrödinger equation is of the same form, namely $t^{-\frac{1}{2}}e^{\frac{i|\cdot|^2}{t}}$. As such, an alternative natural generalisation of the Schrödinger maximal operator is the sequence of maximal operators

$$T_a^* f(x) := \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} t^{-\frac{1}{a}} e^{\frac{i|y|^a}{t}} f(x-y)\, dy \right|$$

for $a > 1$. Letting ϕ denote a positive, even $C^\infty(\mathbb{R})$ function that is supported in $\mathbb{R}\backslash[-\frac{1}{4}, \frac{1}{4}]$ and equal to 1 in $\mathbb{R}\backslash[-\frac{1}{2}, \frac{1}{2}]$, these convolution operators will be seen to be closely related to the sequence of maximal multiplier operators,

$$\widetilde{T}_a^* f(x) := \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \widehat{f}(\xi)\phi(t^{\frac{a-1}{a}}\xi)t^{-\frac{a-2}{2a}}|\xi|^{-\frac{a-2}{2(a-1)}} e^{it|\xi|^{\frac{a}{a-1}}+ix\xi}\, d\xi \right|$$

for $a > 1$, and it is these operators that will be the focus of this chapter. It is noted that by rescaling the temporal variable, it is clear that it is equivalent to write $\widetilde{T}_a^* f(x)$ as

$$\sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \widehat{f}(\xi)\phi(t\xi)|t\xi|^{-\frac{a-2}{2(a-1)}} e^{i|t\xi|^{\frac{a}{a-1}}+ix\xi}\, d\xi \right|$$

and both of these two forms will be used in what follows, as appropriate.

These operators were considered by Luis Vega in his doctoral thesis from 1988[143], where the following was shown:

Theorem 7.1.1 (Vega, 1988) *(i) If $a \in (1,2]$ then*

$$\|\widetilde{T}_a^* f\|_{L^2([-1,1])} \lesssim \|f\|_{H^s(\mathbb{R})}$$

if and only if $s \geqslant \frac{1}{2(a-1)} - \frac{1}{4}$;

(ii) If $a \in [2,4]$ then

$$\|\widetilde{T}_a^* f\|_{L^2([-1,1])} \lesssim \|f\|_{H^s(\mathbb{R})}$$

if $s > \frac{1}{a} - \frac{1}{4}$ and only if $s \geqslant \frac{1}{a} - \frac{1}{4}$;

(iii) If $a \in (4,\infty)$ then

$$\|\widetilde{T}_a^* f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}.$$

Here, the following is proved, extending Vega's results in the case of $a \in (1,4]$ to the global setting:

Theorem 7.1.2 *For any $a \in (1,4]$ and $s \geqslant 0$, the estimate*

$$\|\widetilde{T}_a^* f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

holds for all $f \in H^s(\mathbb{R})$ *if* $s > \frac{3}{4(a-1)} - \frac{1}{4}$ *and only if* $s \geqslant \frac{3}{4(a-1)} - \frac{1}{4}$*. Further for* $a > 4$*,*

$$\|\widetilde{T}_a^* f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}.$$

The relation between the operators T_a^* and $\widetilde{T}_a^*$ will be discussed in Section 7.2, after which the proof of the positive part of this theorem, which is essentially a combination of results of Sjölin and Vega, is presented in Section 7.3. As in Chapter 6, the proof of the negative part depends on a generalisation of the counterexample of Dahlberg and Kenig and is presented in Section 7.4. The method of construction of this generalisation involves solving a non-linear optimisation problem and is described in Section 7.5. This discussion is also relevant to the counterexample of Chapter 6, which was constructed by the same method, and it is also shown that the same approach can reproduce the negative parts of Vega's Theorem 7.1.1. This chapter is concluded with a generalisation of Theorems 7.1.1 and 7.1.2 to the context of complex time, similar to that considered in Chapter 6.

7.2 The Relationship Between the Kernel and Multiplier Maximal Operators

To consider the relationship between T_a^* and $\widetilde{T}_a^*$, observe that for any $f \in \mathscr{S}(\mathbb{R})$,

$$T_a^* f = \sup_{t\in(0,1)} |t^{-1} T_a f(t^{-1}\cdot)| \quad \text{and} \quad \widetilde{T}_a^* f = \sup_{t\in(0,1)} |t^{-1} \widetilde{T}_a f(t^{-1}\cdot)|$$

where

$$T_a f(x) = \int_{\mathbb{R}} e^{i|y|^a} f(x-y)\,dy \quad \text{and} \quad \widetilde{T}_a f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\phi(\xi)|\xi|^{-\frac{a-2}{2(a-1)}} e^{i|\xi|^{\frac{a}{a-1}}+ix\xi}\,d\xi.$$

To establish the extent to which the convolution operator T_a and the multiplier* operator $\widetilde{T}_a$ are related, it will suffice to compare the inverse Fourier transform of the multiplier

*Up to a rescaling in x.

$\phi(\xi)|\xi|^{-\frac{a-2}{2(a-1)}} e^{i|\xi|^{\frac{a}{a-1}}}$ with the convolution kernel $e^{i|x|^a}$, or, conversely, to compare this multiplier with the Fourier transform of the convolution kernel.

The close relationship between these functions is a manifestation of what Elias Stein calls the principle of "duality of phases" in [136]. Indeed on p. 358 he writes that "the Fourier transform of $e^{i\psi(x)}a(x)$ is essentially of the form $e^{-i\widetilde{\psi}(\xi)}a^*(\xi)$, where the pair $(\psi,\widetilde{\psi})$ of phases are 'dual' to each other."* He further claims a particular asymptotic formula of relevance to the current setting, although the discussion is non-rigorous and whilst he states that this principle is a consequence of stationary phase estimates for oscillatory integrals, he is not explicit about the details.

A more rigorous treatment of this matter can be found in a paper by Akihiko Miyachi from 1980, [105]. As a special case of case (ii) of his Lemma 4, the leading order term of an asymptotic expansion for the Fourier transform of the multiplier $\phi(\xi)|\xi|^{-\frac{a-2}{2(a-1)}} e^{i|\xi|^{\frac{a}{a-1}}}$ is given to be essentially the above convolution kernel. However, the following more precise statement is an immediate consequence of case (ii) of his more specialised Lemma 6:

Proposition 7.2.1 *Let $\phi \in C^\infty(\mathbb{R})$ be positive, even, supported in $\mathbb{R}\backslash[-\frac{1}{4},\frac{1}{4}]$ and equal to 1 in $\mathbb{R}\backslash[-\frac{1}{2},\frac{1}{2}]$. Then for $a>1$, there exist non-zero constants $C_a \in \mathbb{C}$ and $c_a \in \mathbb{R}$ such that for $|x| \geqslant 1$,*

$$\int_{\mathbb{R}} e^{i(|\xi|^{\frac{a}{a-1}}+2\pi x\xi)}\phi(\xi)|\xi|^{-\frac{a-2}{2(a-1)}}\,d\xi = (C_a + e_a(x))e^{ic_a|x|^a} + E_a(x),$$

where $|E_a(x)| \lesssim |x|^{-k}$ for all $k>0$ and $\lim_{x\to\pm\infty} e_a(x) = 0$.

In other words, less an error term of the form $e_a(x)e^{ic_a|x|^a} + E_a(x)$, the inverse Fourier transform of the multiplier of the operator $\widetilde{T}_a$ is, up to a constant multiple and a re-scaling, equal to the convolution kernel of the operator T_a. These two operators, and hence also their corresponding maximal operators, $\widetilde{T}_a^*$ and T_a^*, are thus intimately related.

*The choice of symbols has been changed from Stein's original text here to avoid notational conflict.

Unfortunately, the error term in the above proposition is not shown to have quite enough decay to allow equivalence of the boundedness of these two maximal operators to be established. Indeed, if it could be shown for all $x \in \mathbb{R}$ that the inverse Fourier transform of $e^{i|\xi|^{\frac{a}{a-1}}}\phi(\xi)|\xi|^{-\frac{a-2}{2(a-1)}}$ is equal to a constant multiple of

$$e^{ic_a|\cdot|^a} + K$$

where $K \in L^1(\mathbb{R})$ is bounded by an integrable, radially decreasing function, then it would follow that for $f \in \mathscr{S}(\mathbb{R})$ and for each $x \in \mathbb{R}$,

$$\widetilde{T}_a f(x) \approx T_a(f(c_a^{-\frac{1}{a}}\cdot))(c_a^{\frac{1}{a}}x) + K * f(x).$$

Defining $K_t = t^{-1}K(t^{-1}\cdot)$ for each $t \in (0,1)$, given the properties of K, it could then be established that $\sup_{t\in(0,1)} |K_t * f(x)| \lesssim Mf(x)$, where M is the Hardy–Littlewood maximal operator (see, for example, [66, Thm. 2.1.10, p. 82]). This would lead to the conclusion that, between spaces on which M is bounded, boundedness of T_a^* and boundedness of $\widetilde{T}_a^*$ are equivalent. Whilst the above pointwise equivalence can be established for small x using an integration by parts argument, the fact that the term e_a in Proposition 7.2.1 may not be in $L^1(\mathbb{R})$ prevents it from being established for all $x \in \mathbb{R}$.

It should be remarked that in [143], Vega claims to establish this equivalence and whilst his proof of Theorem 7.1.1 is for the operators $\widetilde{T}_a^*$, he states the result for the operators T_a^* via this claimed equivalence. Unfortunately, his argument appears to be erroneous and the author has been unable to repair it.

7.3 Boundedness for Regularity Above the Critical Index

The method used to establish the positive part of Theorem 7.1.2 here is exactly the same as that used by Vega to establish the local results, although in this case a result of Sjölin will also be drawn upon. A theorem originating in a 1956 paper of Stein[134] (see also [15, Sec. 4.3] and [137, Sec. V.4]) on interpolation for families of operators will be required. This in turn requires a definition:

Definition (Admissible Family of Operators) *Let (X,μ) and (Y,ν) be measure spaces and define $S := \{z \in \mathbb{C} : \mathrm{Re}(z) \in [0,1]\}$. Let T_z be a family of linear operators depending on $z \in S$ and mapping from the space of simple functions on X into $\mathcal{M}(Y,\nu)$. Then the family T_z is said to be admissible if for all simple functions f on X and simple functions g on Y, the mapping*

$$z \mapsto \int_Y (T_z f) g \, d\nu$$

is analytic on the interior of S, continuous on S and there exists a constant $a < \pi$ such that

$$e^{-a|\mathrm{Im}(z)|} \log \left| \int_Y (T_z f) g \, d\nu \right|$$

is uniformly bounded in S.

Stein's interpolation theorem can now be stated as follows:

Theorem (Stein, 1956) *Let (X,μ) and (Y,ν) be measure spaces, define $S := \{z \in \mathbb{C} : \mathrm{Re}(z) \in [0,1]\}$ and let T_z be an admissible family of linear operators depending on $z \in S$ and mapping from the space of simple functions on X into $\mathcal{M}(Y,\nu)$. Suppose that for both $j = 0$ and $j = 1$, there exist p_j and $q_j \in [1,\infty]$ and a function $M_j : \mathbb{R} \to \mathbb{R}^+$ that satisfies the bound*

$$\sup_{y \in \mathbb{R}} e^{-b|y|} \log M_j(y) < \infty$$

for some $b < \pi$ such that for each $y \in \mathbb{R}$,

$$\|T_{j+iy} f\|_{L^{q_j}(Y)} \leqslant M_j(y) \|f\|_{L^{p_j}(X)}$$

for all simple functions f on X. Then for any $\theta \in [0,1]$,

$$\|T_\theta f\|_{L^{q_\theta}(Y)} \lesssim \|f\|_{L^{p_\theta}(X)}$$

for all simple functions f on X (and hence by density, for all functions $f \in L^{p_\theta}(X)$) where $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$ and $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$.

To begin the proof of the positive part of Theorem 7.1.2, define for each A, $B \in \mathbb{R}$, $t \in (0,1)$ and for functions $f \in \mathscr{S}(\mathbb{R})$,

$$\widetilde{T}^t_{A,B} f(x) := \left| \int_{\mathbb{R}} \widehat{f}(\xi)\phi(t^{\frac{1}{A}}\xi)t^{-\frac{B}{A}}|\xi|^{-B} e^{it|\xi|^A + ix\xi}\, d\xi \right|,$$

where $\phi \in C^\infty(\mathbb{R})$ is positive, even, supported in $\mathbb{R}\backslash[-\frac{1}{4},\frac{1}{4}]$ and equal to 1 in $\mathbb{R}\backslash[-\frac{1}{2},\frac{1}{2}]$. Let $\widetilde{T}^*_{A,B}$ denote the maximal operator, $\sup_{t\in(0,1)} |\widetilde{T}^t_{A,B}\cdot|$. These operators coincide with the operators $\widetilde{T}^*_a$ in the case of $A = \dfrac{a}{a-1}$, $B = \dfrac{a-2}{2(a-1)}$.

Consider first that for such choices of A and B, when $a \in (1,2]$, it is the case that $B \leqslant 0$, so for each $x \in \mathbb{R}$ and $t \in (0,1)$,

$$\begin{aligned} |\widetilde{T}^t_{A,B} f(x)| &= |t|^{-\frac{B}{A}} \left| \int_{\mathbb{R}} \widehat{f}(\xi)\phi(t^{\frac{1}{A}}\xi)|\xi|^{-B} e^{it|\xi|^A + ix\xi}\, d\xi \right| \\ &\leqslant |\widetilde{T}^t_{A,0} f_B(x)|, \end{aligned}$$

where $\widehat{f_B} = |\cdot|^{-B}\widehat{f}$.

As was already mentioned in Section 6.1, Sjölin established in 1994 in [130] that for $a > 1$, the operators

$$S^*_a f(x) := \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \widehat{f}(\xi) e^{it|\xi|^a} e^{ix\xi}\, d\xi \right|$$

are bounded from $H^s(\mathbb{R})$ into $L^2(\mathbb{R})$ if $s > \frac{a}{4}$ and only if $s \geqslant \frac{a}{4}$. Now, Sjölin's proof is equally applicable to the operator $\sup_{t\in(0,1)} |\widetilde{T}^t_{A,0}\cdot|$; indeed, the insertion of the additional term, $\phi(t^{\frac{1}{a}}\xi)$, into the definition of S^*_a serves only to simplify the oscillatory integral estimates that Sjölin carries out in [130].* Assuming that $\|f_B\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-B}(\mathbb{R})}$, it follows that

$$\|T^*_a f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

*For example, the introduction of a cut-off function around the origin in the definition of $p_{N,M,\varepsilon}$ on p. 108 is no longer required. Some of the terms of the decomposition of the oscillatory integral being bounded may disappear entirely; the finite number of terms that are affected by the new term that do not disappear are made no more difficult to bound by the presence of an additional smooth cut-off function.

if $s > \frac{A}{4} - B$ and only if $s \geqslant \frac{A}{4} - B$. Since $A = \frac{a}{a-1}$ and $B = \frac{a-2}{2(a-1)}$, it can be seen that $\frac{A}{4} - B = \frac{3}{4(a-1)} - \frac{1}{4}$ and so the positive part of Theorem 7.1.2 is proved for $a \in (1,2]$.

To prove that $\|f_B\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-B}(\mathbb{R})}$, first note that $\|f_B\|_{\dot{H}^r(\mathbb{R})} = \|f\|_{\dot{H}^{r-B}(\mathbb{R})}$ for each $r \geqslant 0$. Now, since for functions $g \in H^s(\mathbb{R})$ it is the case that $\|g\|_{H^s(\mathbb{R})} \sim \|g\|_{L^2(\mathbb{R})} + \|g\|_{\dot{H}^s(\mathbb{R})}$, using that $B < 0$, it follows that

$$\begin{aligned}
\|f_B\|_{H^s(\mathbb{R})} &\lesssim \|f_B\|_{L^2(\mathbb{R})} + \|f_B\|_{\dot{H}^s(\mathbb{R})} \\
&= \|f\|_{\dot{H}^{-B}(\mathbb{R})} + \|f\|_{\dot{H}^{s-B}(\mathbb{R})} \\
&\lesssim \|f\|_{L^2(\mathbb{R})} + \|f\|_{H^{-B}(\mathbb{R})} + \|f\|_{H^{s-B}(\mathbb{R})} \\
&\leqslant 3\|f\|_{H^{s-B}(\mathbb{R})}
\end{aligned}$$

and so the inequality is established.

The positive part of Theorem 7.1.2 for $a \in (4,\infty)$ was already established by Vega and stated in Theorem 7.1.1. To prove the remaining case of $a \in (2,4]$ it will also be necessary to know that Vega established the $a > 4$ result as a consequence of the following more general result on the operators $\widetilde{T}^t_{A,B}$:[143, Thm. 1.15, pp. 35-36]

Theorem 7.3.1 (Vega) *For $A > 1$, $B > \frac{A}{4}$ and $f \in \mathscr{S}(\mathbb{R})$,*

$$\|\sup_{t\in\mathbb{R}^+} |\widetilde{T}^t_{A,B} f|\|_{L^2(\mathbb{R})} \lesssim_{A,B} \|f\|_{L^2(\mathbb{R})}.$$

The positive result for $a \in (2,4]$ now follows as a consequence of interpolation between this result and Sjölin's result, using Stein's theorem. Indeed, observe that, by adapting the operators considered, Stein's theorem may easily be modified to interpolate between an estimate from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and an estimate from $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$. As such, fixing $A > 1$ and interpolating between Sjölin's result (considered as a result on boundedness of $\widetilde{T}^*_{A,0}$)

and Vega's result with B considered as the interpolation parameter, it follows that for each $\theta \in (0,1)$,

$$\|\widetilde{T}^*_{A,\theta\frac{A}{4}} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for $s > \frac{A}{4}(1-\theta)$.* Choosing $A = \frac{a}{a-1}$ and $B = \frac{a-2}{2(a-1)}$, it follows that $B = 1 - \frac{A}{2}$, so for $\theta\frac{A}{4} = B$, it must be the case that $\theta = 2 - \frac{4}{a}$. As such, $\frac{A}{4}(1-\theta) = \frac{3}{4(a-1)} - \frac{1}{4}$, so the desired result is established.

Given that the thesis of Vega is not a widely available resource, for completeness, a proof of Theorem 7.3.1 based on that from [143] will now be provided. As in the proof of Theorem 6.1.2 in Section 6.2, it depends on the Kolmogorov–Seliverstov–Plessner method and an oscillatory integral estimate. The latter can be stated as the following lemma:

Lemma 7.3.2 *Let $A > 1$, $t \in [-2,2]$ and $x \in \mathbb{R}$. Let $\psi \in \mathscr{S}(\mathbb{R})$ be supported in $(-2,-\frac{1}{8}) \cup (\frac{1}{8},2)$ and for each $k \in \mathbb{N} \cup \{0\}$, define $\psi_k := \psi(2^{-k}\cdot)$. Then there exists $K \in L^1(\mathbb{R})$ such that for all $k \in \mathbb{N} \cup \{0\}$ and all $x \in \mathbb{R}$,*

$$2^{-\frac{kA}{2}}\left|\int_{\mathbb{R}} e^{i(t|\xi|^A + x\xi)}\psi_k(\xi)\,d\xi\right| \leqslant K(x).$$

Proof The proof of this lemma will be similar in nature to the proof of Lemma 6.2.1 in Section 6.2. Again, the key will be isolating a neighbourhood around a point of possible stationary phase, $\rho := (\frac{|x|}{tA})^{\frac{1}{A-1}}$, and applying Van der Corput's Lemma. It will be assumed here without loss of generality that $\xi > 0$ and $x < 0$ and for simplicity, it will be assumed that $\rho \in (2^{k-3}, 2^{k+1})$ (the interval in which ψ_k is supported); the proof is otherwise more straightforward.

---

*That the operator $\widetilde{T}^*_{A,B}$ satisfies the hypotheses of Stein's theorem in parameter B can be seen by noting that for any suitable function f and $y \in \mathbb{R}$, $\widetilde{T}^*_{A,B+iy} f = \widetilde{T}^*_{A,B} g_y$ where $\widehat{g_y} = \widehat{f} e^{-iy\log|\cdot|}$.

Define $F(\xi) := t|\xi|^A + x\xi$ and choose $\phi_2 \in \mathscr{S}(\mathbb{R})$ to be supported in $\{\xi \in (2^{k-3}, 2^{k+1}) : |F'(\xi)| \leqslant \frac{|x|}{2}\}$ and equal to 1 in $\{\xi \in (2^{k-3}, 2^{k+1}) : |F'(\xi)| \leqslant \frac{|x|}{4}\}$. Define $\phi_1 := (1-\phi_2)\chi_{(2^{k-3},\rho)}$ and $\phi_3 := (1-\phi_2)\chi_{(\rho,2^{k+1})}$, and for each $j \in \{1,2,3\}$, define

$$J_j := \int_{\mathbb{R}} e^{i(t|\xi|^A + x\xi)} \psi_k(\xi)\phi_j \, d\xi.$$

As previously, J_2 constitutes the integral near the point of stationary phase, ρ, with J_1 and J_3 constituting the remaining parts of the integral on either side of this neighbourhood.

For $j \in \{1,3\}$, note first that it is clear from the size of the support of ψ_k that $|J_j|$ can be bounded by a constant multiple of 2^k. In particular, it certainly follows that

$$\|2^{-\frac{kA}{2}} J_j\|_{L^1([-2^{-k},0])} \lesssim 2^{-\frac{kA}{2}} \lesssim 1.$$

Also, for $x < -2^{-k}$, using that $F'(\xi) \gtrsim |x|$ and integrating by parts twice, exactly as in the bounds on J_1 and J_3 in Section 6.2.2 (the proof of Lemma 6.2.1 for large x), it can be seen that $|J_j| \lesssim 2^{-k}|x|^{-2}$, so

$$\|2^{-\frac{kA}{2}} J_j\|_{L^1((-\infty,-2^{-k}])} \lesssim 2^{-\frac{kA}{2}} \lesssim 1.$$

The bound on J_j for $j \in \{1,3\}$ is thus complete.

For J_2, note that since $\xi \sim \rho$ and $\xi \sim 2^k$ in the range of integration, it is the case that $\rho \sim 2^k$. Using that $t \in [-2,2]$, it follows that $|x| \lesssim 2^{k(A-1)}$ and $|t| \gtrsim |x|2^{k(1-A)}$. Since $F''(\xi) = A(A-1)t|\xi|^{A-2}$, it follows that

$$|F''(\xi)| \gtrsim |x|2^{k(1-A)}2^{k(A-2)} = |x|2^{-k}.$$

As such, applying Van der Corput's Lemma with the second derivative,

$$2^{-k\frac{A}{2}}|J_2| \lesssim 2^{-k\frac{A}{2}} 2^{\frac{k}{2}} |x|^{-\frac{1}{2}}.$$

Since $|x| \lesssim 2^{k(A-1)}$, by integrating the above expression, it follows that the L^1 norm of $2^{-k\frac{A}{2}} J_2$ can be bounded by a constant and so the proof of the lemma is complete. $\square$

Given this lemma, Theorem 7.3.1 can be proved as follows:

Proof (Theorem 7.3.1) Fix $A > 1$ and $B > \frac{A}{4}$. Choose a function $\psi \in \mathscr{S}(\mathbb{R})$ supported on $[-2,-\frac{1}{2}] \cup [\frac{1}{2},2]$ and a function $\psi_0 \in \mathscr{S}(\mathbb{R})$ supported on $[-2,2]$ such that

$$\sum_{k=0}^{\infty} \psi_k(\xi) = 1$$

for all $\xi \in \mathbb{R}$, where $\psi_k := \psi(2^{-k}\cdot)$ for $k \in \mathbb{N}$. For each $k \in \mathbb{N} \cup \{0\}$, and each $t \in \mathbb{R}^+$, define

$$\widetilde{T}^t_{A,B,k} f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\phi(t\xi)t^{-B}|\xi|^{-B} e^{i(t^A|\xi|^A + x\xi)} \psi_k(t\xi)\, d\xi.$$

To prove Theorem 7.3.1, it will certainly suffice to show that

$$\| \sup_{t\in\mathbb{R}^+} |\widetilde{T}^t_{A,B,k} f| \|_{L^2(\mathbb{R})} \lesssim 2^{-\frac{k}{2}(B-\frac{A}{4})} \|f\|_{L^2(\mathbb{R})}.$$

It is further claimed that this will follow from the locally maximal bound

$$\| \sup_{t\in(1,2)} |\widetilde{T}^t_{A,B,k} f| \|_{L^2(\mathbb{R})} \lesssim 2^{-\frac{k}{2}(B-\frac{A}{4})} \|f\|_{L^2(\mathbb{R})}.$$

To see this, assuming that this bound is true, write that

$$\int_{\mathbb{R}} \sup_{t\in\mathbb{R}^+} |\widetilde{T}^t_{A,B,k} f(x)|^2\, dx \leqslant \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \sup_{t\in[2^j,2^{j+1}]} |\widetilde{T}^t_{A,B,k} f(x)|^2\, dx.$$

Given that $\operatorname{supp}(\psi_k) \subseteq [-2^{k+1},-2^{k-1}] \cup [2^{k-1},2^{k+1}]$ for each $k \in \mathbb{N}$, for $t \in [2^j,2^{j+1}]$, $\psi_k(t\xi)$ is non-zero only if $\xi \in [-2^{k-j+1},-2^{k-j-2}] \cup [2^{k-j-2},2^{k-j+1}]$. Since $\operatorname{supp}(\phi\psi_0) \subseteq [-2,-\frac{1}{4}] \cup [\frac{1}{4},2]$, it can be assumed for all $k \in \mathbb{N} \cup \{0\}$ that $\xi \in [-2^{k-j+1},-2^{k-j-3}] \cup [2^{k-j-3},2^{k-j+1}]$ in the integrand defining $\widetilde{T}^t_{A,B,k}$. Consequently, letting $f_{j,k}$ be the function such that

$$\widehat{f_{j,k}} = \chi_{\pm[2^{k-j-3},2^{k-j+1}]} \widehat{f},$$

the above quantity can be seen to be equal to

$$\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\sup_{t\in[2^j,2^{j+1}]}|\widetilde{T}^t_{A,B,k}f_{j,k}(x)|^2\,dx.$$

Now, by a change of variables,

$$\widetilde{T}^t_{A,B,k}f(x)=\int_{\mathbb{R}}2^{-j}\widehat{f}(2^{-j}\xi)\phi((2^{-j}t)\xi)((2^{-j}t)|\xi|)^{-B}e^{i((2^{-j}t)^A|\xi|^A+(2^{-j}x)\xi)}\psi_k((2^{-j}t)\xi)\,d\xi,$$

so

$$\begin{aligned}\int_{\mathbb{R}}\sup_{t\in\mathbb{R}^+}|\widetilde{T}^t_{A,B,k}f(x)|^2\,dx &\leqslant \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}\sup_{t\in(1,2)}|\widetilde{T}^t_{A,B,k}(f_{j,k}(2^j\cdot))(2^{-j}x)|^2\,dx\\ &\lesssim \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}}|f_{j,k}(x)|^2\,dx\\ &\lesssim \int_{\mathbb{R}}|f(x)|^2\,dx\end{aligned}$$

by “almost orthogonality” of the $f_{j,k}$ (that is that for any $j\in\mathbb{Z}$, $\text{supp}(\widehat{f_{j,k}})\cap\text{supp}(\widehat{f_{j',k}})$ has positive measure for at most seven choices of $j'\in\mathbb{Z}$).

It remains to show that

$$\|\sup_{t\in(1,2)}|\widetilde{T}^t_{A,B,k}f|\|_{L^2(\mathbb{R})}\lesssim 2^{-\frac{k}{2}(B-\frac{A}{4})}\|f\|_{L^2(\mathbb{R})}$$

for all $k\in\mathbb{N}\cup\{0\}$. Dualising and linearising the supremum in the desired bound, it can be seen that it suffices to show that for any $g\in L^2(\mathbb{R})$ with $\|g\|_{L^2(\mathbb{R})}=1$ and any measurable function, $t:\mathbb{R}\to(1,2)$,

$$\int_{\mathbb{R}}\widetilde{T}^{t(x)}_{A,B,k}f(x)\overline{g(x)}\,dx\lesssim\|f\|_{L^2(\mathbb{R})}$$

with constant independent of the function t.

By Fubini's theorem

$$\begin{aligned}&\int_{\mathbb{R}} \widetilde{T}^{t(x)}_{A,B,k} f(x)\overline{g(x)}\,dx\\ =\;&\int_{\mathbb{R}} \widehat{f}(\xi)|\xi|^{-B}\left(\int_{\mathbb{R}} \phi(t(x)\xi)e^{i(t^A(x)|\xi|^A+x\xi)}t^{-B}(x)\psi_k(t(x)\xi)\overline{g(x)}\,dx\right)d\xi.\end{aligned}$$

Since $\operatorname{supp}(\phi\psi_k) \subseteq [-2^{k+1}, -2^{k-2}] \cup [2^{k-2}, 2^{k+1}]$ and $t(x) \in (1,2)$ for all $x \in \mathbb{R}$, it can be assumed that $|\xi| \gtrsim 2^k$ in this integral, so by the Cauchy–Schwarz inequality, the above is bounded by a constant multiple of

$$2^{-kB}\|f\|_{L^2(\mathbb{R})}\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} \phi(t(x)\xi)e^{i(t^A(x)|\xi|^a+x\xi)}t^{-B}(x)\psi_k(t(x)\xi)\overline{g(x)}\,dx\right|^2 d\xi\right)^{\frac{1}{2}}.$$

Define

$$I_k := 2^{-\frac{kA}{2}}\int_{\mathbb{R}}\left|\int_{\mathbb{R}} \phi(t(x)\xi)e^{i(t^A(x)|\xi|^A+x\xi)}t^{-B}(x)\psi_k(t(x)\xi)\overline{g(x)}\,dx\right|^2 d\xi.$$

Then

$$\int_{\mathbb{R}} \widetilde{T}^{t(x)}_{A,B,k} f(x)\overline{g(x)}\,dx \lesssim 2^{-k(B-\frac{A}{4})}\|f\|_{L^2(\mathbb{R})} I_k^{\frac{1}{2}},$$

so to complete the proof of Theorem 7.3.1, it suffices to show that I_k is uniformly bounded. However, writing that $\widetilde{\psi_k} := \phi\psi_k$, by another application of Fubini's theorem,

$$\begin{aligned}I_k \;=\;& \int_{\mathbb{R}}\int_{\mathbb{R}} t^{-B}(x)t^{-B}(y)\overline{g(x)}g(y)\left(2^{-\frac{kA}{2}}\int_{\mathbb{R}} e^{i((t^A(x)-t^A(y))|\xi|^A+(x-y)\xi)}\widetilde{\psi_k}(t(x)\xi)\right.\\ &\left.\times\widetilde{\psi_k}(t(y)\xi)\,d\xi\right)dy\,dx.\end{aligned}$$

The integral in ξ here falls into the scope of Lemma 7.3.2, so using that $t^{-B}(x), t^{-B}(y) \leqslant 1$, it follows that there exists a function $K \in L^1(\mathbb{R})$, independent of k, such that

$$I_k \leqslant \int_{\mathbb{R}} |g(x)|(|g| * |K|)(x)\,dx.$$

By the Cauchy–Schwarz inequality and Young's convolution inequality, it follows that I_k is uniformly bounded and thus the proof of Theorem 7.3.1 is complete. □

7.4 Failure of Boundedness for Regularity Below the Critical Index

As in Section 6.3, a generalisation of the counterexample of Dahlberg and Kenig will be presented here to show that for $a \in (1,4)$, the bound

$$\|\widetilde{T}_a^* f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

fails for any $s < \frac{3}{4(a-1)} - \frac{1}{4}$. Following the same reasoning as there, it will suffice to show that there exist functions f_v for $v \in (0, v_0)$ for some small v_0, the $H^s(\mathbb{R})$ norms of which tend to zero as $v \to 0$ for $s < \frac{3}{4(a-1)} - \frac{1}{4}$, and corresponding functions $t_v : \mathbb{R} \to (0,1)$, such that the $L^2(\mathbb{R})$ norms of the functions

$$R_a^{t_v(x)} f_v(x) := \left| \int_{\mathbb{R}} \widehat{f}(\xi)\phi(t_v(x)\xi)|t_v(x)\xi|^{-\frac{a-2}{2(a-1)}} e^{i|t_v(x)\xi|^{\frac{a}{a-1}}+ix\xi}\, d\xi \right|$$

are uniformly bounded below in v, where $\phi \in C^\infty(\mathbb{R})$ is positive, even, supported in $\mathbb{R}\backslash[-\frac{1}{4},\frac{1}{4}]$ and equal to 1 in $\mathbb{R}\backslash[-\frac{1}{2},\frac{1}{2}]$.

Let $g_v \in \mathscr{S}(\mathbb{R})$ be supported on $[-v^{3-a}, v^{3-a}]$ and equal to 1 on $[-\frac{1}{2}v^{3-a}, \frac{1}{2}v^{3-a}]$. Define $\widehat{f}_v(\xi) = v g_v(v\xi + \frac{1}{v^{2a-3}})$ and observe that

$$\begin{aligned}
\|f_v\|_{\dot{H}^s(\mathbb{R})}^2 &= v^2 \int_{\mathbb{R}} \left| g_v\left(v\xi + \frac{1}{v^{2a-3}}\right)\right|^2 |\xi|^{2s}\, d\xi \\
&= v^{1-2s} \int_{\mathbb{R}} \left| g_v\left(\xi + \frac{1}{v^{2a-3}}\right)\right|^2 |\xi|^{2s}\, d\xi.
\end{aligned}$$

For $g_v(\xi + \frac{1}{v^{2a-3}}) \neq 0$, it must be the case that $\xi + \frac{1}{v^{2a-3}} \in [-\frac{1}{v^{a-3}}, \frac{1}{v^{a-3}}]$, so necessarily, $|\xi| \lesssim \frac{1}{v^{2a-3}}$. It thus follows that

$$\begin{aligned}
\|f_v\|_{\dot{H}^s(\mathbb{R})}^2 &\lesssim v^{1-2s} v^{2s(3-2a)} v^{3-a} \\
&= v^{4-a-4s(a-1)}.
\end{aligned}$$

As in Section 6.3, given that $\|f_v\|_{H^s(\mathbb{R})} \sim \|f_v\|_{L^2(\mathbb{R})} + \|f_v\|_{\dot{H}^s(\mathbb{R})}$ and the fact that the above argument also shows that $\|f_v\|_{L^2(\mathbb{R})} \to 0$ as $v \to 0$, it follows that $\|f_v\|_{H^s(\mathbb{R})} \to 0$ as $v \to 0$ whenever $s < \frac{3}{4(a-1)} - \frac{1}{4}$.

Consider that

$$R_a^t f_v(x) = \int_{\mathbb{R}} v g_v\Big(v\xi + \frac{1}{v^{2a-3}}\Big)\phi(t\xi)|t\xi|^{-\frac{a-2}{2(a-1)}} e^{i|t\xi|^{\frac{a}{a-1}}} e^{ix\xi}\, d\xi$$

and substitute $\eta = v\xi + \frac{1}{v^{2a-3}}$ to obtain (after removal and addition of unimodular factors in the integrand that do not depend on the variable of integration) that $|R_a^t f_v(x)|$ is equal to

$$\left|\int_{-v^{3-a}}^{v^{3-a}} g_v(\eta)\phi\Big(t\Big(\frac{\eta}{v} - \frac{1}{v^{2a-2}}\Big)\Big)\Big|t\Big(\frac{\eta}{v} - \frac{1}{v^{2a-2}}\Big)\Big|^{-\frac{a-2}{2(a-1)}} e^{i|t(\frac{\eta}{v}-\frac{1}{v^{2a-2}})|^{\frac{a}{a-1}}} e^{i(x\frac{\eta}{v} - \frac{t^{\frac{a}{a-1}}}{v^{2a}})}\, d\eta\right|.$$

Fix $x = cv^{-2}$ for some small positive constant c and set $t = x^{1-\frac{1}{a}} v^{2-\frac{2}{a}}(1-\frac{1}{a})^{1-\frac{1}{a}}$. Noting that $t \sim 1$ as $v \to 0$, for $\eta \in [-v^{3-a}, v^{3-a}]$, the quantity $t(\frac{\eta}{v} - \frac{1}{v^{2a-2}})$ is large and negative. Consequently, it may be assumed that

$$\phi\Big(t\Big(\frac{\eta}{v} - \frac{1}{v^{2a-2}}\Big)\Big) \equiv 1.$$

Now,

$$t^{\frac{a}{a-1}}\Big(\frac{1}{v^{2a-2}} - \frac{\eta}{v}\Big)^{\frac{a}{a-1}} = t^{\frac{a}{a-1}}\left(\frac{1}{v^{2a}} - \frac{(\frac{a}{a-1})\eta}{v^3} + O\Big(\frac{\eta^2}{v^{6-2a}}\Big)\right),$$

so given the above choice of t,

$$\begin{aligned}\Big|t\Big(\frac{\eta}{v} - \frac{1}{v^{2a-2}}\Big)\Big|^{\frac{a}{a-1}} + \frac{x\eta}{v} - \frac{t^{\frac{a}{a-1}}}{v^{2a}} &= \frac{x\eta}{v} - \frac{t^{\frac{a}{a-1}}(\frac{a}{a-1})\eta}{v^3} + O\Big(\frac{t^{\frac{a}{a-1}}\eta^2}{v^{6-2a}}\Big)\\ &= O\Big(\frac{xv^2\eta^2}{v^{6-2a}}\Big)\\ &\ll 1.\end{aligned}$$

Bounding the integral below by its real part replaces the exponential terms by a cosine term,

$$\cos\left(\left|t\left(\frac{\eta}{v}-\frac{1}{v^{2a-2}}\right)\right|^{\frac{a}{a-1}}+x\frac{\eta}{v}-\frac{t^{\frac{a}{a-1}}}{v^{2a}}\right),$$

and from the above deductions, this term can be controlled to be essentially constant, so long as v_0 is chosen to be sufficiently small.

Additionally,

$$\begin{aligned}\left|t\left(\frac{\eta}{v}-\frac{1}{v^{2a-2}}\right)\right|^{-\frac{a-2}{2(a-1)}} &\sim t^{-\frac{a-2}{2(a-1)}}v^{2(a-1)\frac{a-2}{2(a-1)}}\\ &\approx x^{-(1-\frac{1}{a})(\frac{a-2}{2(a-1)})}v^{-(2-\frac{2}{a})(\frac{a-2}{2(a-1)})}v^{a-2}\\ &= x^{-\frac{a-2}{2a}}v^{-\frac{a-2}{a}+a-2}\\ &= x^{\frac{1}{a}-\frac{1}{2}}v^{a+\frac{2}{a}-3}.\end{aligned}$$

It follows that $|R_a^t f_v(x)|\gtrsim x^{\frac{1}{a}-\frac{1}{2}}v^{a+\frac{2}{a}-3}v^{3-a}$, so

$$\|\widetilde{T}_a^* f_v\|_{L^2(\mathbb{R})}^2\gtrsim\left(\int_{x\sim v^{-2}}x^{\frac{2}{a}-1}\,dx\right)v^{2a+\frac{4}{a}-6}v^{6-2a}=1$$

and so $\|\widetilde{T}_a^* f_v\|_{L^2(\mathbb{R})}$ is bounded below uniformly in v, as required. The negative part of Theorem 7.1.2 is thus proved.

7.5 Generation of Counterexamples

In this section, details on the construction of the counterexample given in Section 7.4 will be provided. Whilst it is not discussed directly, the counterexample from Section 6.3 was constructed in the same way and so this discussion is equally relevant to Chapter 6.

The idea behind the construction of these counterexamples is to introduce a number of parameters to the argument of Dahlberg and Kenig used to establish failure of boundedness of the Schrödinger maximal operator (as discussed briefly in Section 5.1). As in their work and as has already been seen in Sections 6.3 and 7.4, a counterexample is generated by determining a sequence of functions f_v depending on a parameter $v \in (0, v_0)$, for some small v_0, such that the corresponding images under the maximal operator are bounded below in L^2 norm uniformly in v, but for which it is also the case that $\|f_v\|_{H^s(\mathbb{R})} \to 0$ as $v \to 0$ whenever s is smaller than some $s_0 > 0$. In the process of attempting to follow a scheme similar to theirs, constraints for the various parameters are generated; the problem then becomes to find an optimal set of parameters that satisfy these constraints and maximise s_0, thus maximising the applicability of the counterexample.

To begin with, fix parameters α, β, δ, $\varepsilon \in \mathbb{R}$ with $\alpha > 0$. Let $g_v \in \mathscr{S}(\mathbb{R})$ be supported in $[-v^\delta, v^\delta]$ and equal to 1 in $[-\frac{1}{2}v^\delta, \frac{1}{2}v^\delta]$, and set $\widehat{f}_v(\xi) := v^\alpha g_v(v^\alpha \xi + \frac{1}{v^\beta})$.* Then

$$\begin{aligned} \|f_v\|^2_{\dot{H}^s(\mathbb{R})} &= v^{2\alpha} \int_{\mathbb{R}} \left| g_v\left(v^\alpha \xi + \frac{1}{v^\beta}\right)\right|^2 |\xi|^{2s}\, d\xi \\ &= v^{\alpha(1-2s)} \int_{\mathbb{R}} \left| g_v\left(\xi + \frac{1}{v^\beta}\right)\right|^2 |\xi|^{2s}\, d\xi. \end{aligned}$$

For the integrand here to be non-zero, $\xi + \frac{1}{v^\beta} \in [-v^\delta, v^\delta]$, so $|\xi| \lesssim \frac{1}{v^{\max(\beta,-\delta)}}$. It follows that

$$\|f_v\|^2_{\dot{H}^s(\mathbb{R})} \lesssim v^{\alpha(1-2s)+\delta+2s\min(-\beta,\delta)}$$

and so $\|f_v\|_{\dot{H}^s(\mathbb{R})} \to 0$ as $v \to 0$ for all $s < s_0$ if

$$\boxed{\alpha(1-2s_0) + \delta + 2s_0 \min(-\beta, \delta) \geqslant 0} \tag{1}$$

As before, given that $\|f_v\|_{H^s(\mathbb{R})} \sim \|f_v\|_{L^2(\mathbb{R})} + \|f_v\|_{\dot{H}^s(\mathbb{R})}$ and the fact that the above argument also shows that $\|f_v\|_{L^2(\mathbb{R})} \to 0$ as $v \to 0$, it follows that $\|f_v\|_{H^s(\mathbb{R})} \to 0$ as $v \to 0$ under the same condition.

*By reparameterisation, this is equivalent to translating, dilating and rescaling a bump function g on $[-1,1]$ by writing $\widehat{f}_v(\xi) := v^\delta v^\alpha g(v^\alpha + \frac{1}{v^\beta})$.

For notational convenience, define $A := \frac{a}{a-1}$, $B := \frac{a-2}{2(a-1)}$ and consider that it suffices to consider the sequence of functions,

$$R_a^t f_v(x) = \int_{\mathbb{R}} v^\alpha g_v\left(v^\alpha \xi + \frac{1}{v^\beta}\right)\phi(t\xi)|t\xi|^{-B} e^{i|t\xi|^A} e^{ix\xi}\, d\xi,$$

where $\phi \in C^\infty(\mathbb{R})$ is positive, even, supported in $\mathbb{R}\backslash[-\frac{1}{4},\frac{1}{4}]$ and equal to 1 in $\mathbb{R}\backslash[-\frac{1}{2},\frac{1}{2}]$. By substituting $\eta = v^\alpha \xi + \frac{1}{v^\beta}$ and removing and adding unimodular terms, it follows that $|R_a^t f_v(x)|$ is equal to

$$\left|\int_{-v^\delta}^{v^\delta} g_v(\eta)\phi\left(t\left(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}}\right)\right)\left|t\left(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}}\right)\right|^{-B} e^{i|t(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}})|^A} e^{i(\frac{x\eta}{v^\alpha} - \frac{t^A}{v^{A(\alpha+\beta)}})}\, d\eta\right|.$$

It is desirable that $t\xi$ is large so that ϕ may be assumed to be equal to 1 here. As t must be local, ξ itself must be large, so given the domain of integration here, assume that $\alpha - \delta < \alpha + \beta$, that is

$$\boxed{\beta > -\delta} \tag{2}$$

Now, by the binomial expansion (using constraint (2)),

$$\left|t\left(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}}\right)\right|^A = t^A\left(\frac{1}{v^{A(\alpha+\beta)}} - \frac{\eta}{v^{A\alpha+(A-1)\beta}} + O\left(\frac{\eta^2}{v^{A\alpha+(A-2)\beta}}\right)\right).$$

As such,

$$\left|t\left(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}}\right)\right|^A + \frac{x\eta}{v^\alpha} - \frac{t^A}{v^{A(\alpha+\beta)}} = \frac{x\eta}{v^\alpha} - \frac{t^A\eta}{v^{A\alpha+(A-1)\beta}} + O\left(\frac{t^A\eta^2}{v^{A\alpha+(A-2)\beta}}\right).$$

Fix $x \sim v^\varepsilon$ and choose t so as to eliminate the first two terms, that is

$$t = x^{\frac{1}{A}} v^{(\alpha+\beta)(1-\frac{1}{A})}.$$

In order to ensure that this choice of t is local, it is necessary that $\frac{\varepsilon}{A}+(\alpha+\beta)(1-\frac{1}{A})\geqslant 0$, which, given that $A>1$, is equivalent to

$$\boxed{\varepsilon+(A-1)(\alpha+\beta)\geqslant 0} \tag{3}$$

Given the earlier desire that $t\xi$ must be large, it is also imposed that $\frac{\varepsilon}{A}+(\alpha+\beta)(1-\frac{1}{A})-(\alpha+\beta)\leqslant 0$, that is

$$\boxed{\varepsilon\leqslant\alpha+\beta} \tag{4}$$

Given the choice of t, it follows that

$$\left|t\left(\frac{\eta}{v^{\alpha}}-\frac{1}{v^{\alpha+\beta}}\right)\right|^{A}+\frac{x\eta}{v^{\alpha}}-\frac{t^{A}}{v^{A(\alpha+\beta)}}=O\Big(\frac{xv^{(\alpha+\beta)(A-1)}\eta^{2}}{v^{A\alpha+(A-2)\beta}}\Big)=O\Big(\frac{x\eta^{2}}{v^{\alpha-\beta}}\Big).$$

Bounding the integral below by its real part replaces the exponential terms in the integrand by the cosine term,

$$\cos\left(\left|t\left(\frac{\eta}{v^{\alpha}}-\frac{1}{v^{\alpha+\beta}}\right)\right|^{A}+\frac{x\eta}{v^{\alpha}}-\frac{t^{A}}{v^{A(\alpha+\beta)}}\right),$$

which can, from the above, be controlled to be essentially constant (when v_0 is chosen to be sufficiently small), so long as

$$\boxed{\varepsilon+2\delta-(\alpha-\beta)\geqslant 0} \tag{5}$$

Now, given constraint (2),

$$\begin{aligned}&\left|t\left(\frac{\eta}{v^{\alpha}}-\frac{1}{v^{\alpha+\beta}}\right)\right|^{-B}\\ \sim\quad & x^{-\frac{B}{A}}v^{(\alpha+\beta)(\frac{B}{A}-B)}v^{B(\alpha+\beta)}\end{aligned}$$

$$= \quad x^{-\frac{B}{A}} v^{(\alpha+\beta)\frac{B}{A}}.$$

From these observations it follows that $|R_a^t f_v(x)|$ may be bounded below by a constant multiple of the above quantity multiplied by the size of the domain of the integral and so in particular,

$$|\widetilde{T}_a^* f_v(x)| \gtrsim x^{-\frac{B}{A}} v^{(\alpha+\beta)\frac{B}{A}+\delta},$$

so

$$\|\widetilde{T}_a^* f_v\|_{L^2(\mathbb{R})}^2 \gtrsim \left(\int_{x\sim v^\varepsilon} x^{-\frac{2B}{A}}\, dx\right) v^{2(\alpha+\beta)\frac{B}{A}+2\delta} \approx v^{\varepsilon(1-\frac{2B}{A})+2(\alpha+\beta)\frac{B}{A}+2\delta}.$$

Consequently, $\|\widetilde{T}_a^* f_v\|_{L^2(\mathbb{R})}^2 \gtrsim 1$ uniformly in v so long as $\varepsilon\left(1-\frac{2B}{A}\right)+2(\alpha+\beta)\frac{B}{A}+2\delta \leqslant 0$, that is

$$\boxed{\varepsilon(A-2B)+2B(\alpha+\beta)+2A\delta \leqslant 0} \tag{6}$$

So long as all of the above boxed constraints are satisfied, it can be seen that the sequence of functions, f_v, provides a counterexample for the bound $\|\widetilde{T}_a^* f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$ for all $s < s_0$. As such, substituting $A = \frac{a}{a-1}$ and $B = \frac{a-2}{2(a-1)}$, the goal is to maximise $s_0 > 0$ subject to the following constraints:*

$$\alpha(1-2s_0)+\delta-2s_0\beta \geqslant 0; \quad (1) \qquad \beta+\delta > 0; \quad (2)$$

$$(a-1)\varepsilon+\alpha+\beta \geqslant 0; \quad (3) \qquad \varepsilon-\alpha-\beta \leqslant 0; \quad (4)$$

$$\varepsilon+2\delta-(\alpha-\beta) \geqslant 0; \quad (5) \qquad 2\varepsilon+(a-2)(\alpha+\beta)+2a\delta \leqslant 0. \quad (6)$$

*Strictly speaking, it is already known that constraint (2) is redundant. Indeed, given locality of t, the assumption that $t\xi$ is large which led to constraint (4) is at least as strong as the assumption that ξ is large, which led to constraint (2). Nonetheless, the aim of this section is to demonstrate a general method that might be applicable to a wider range of problems, so for the sake of clarity, this constraint is not removed.

Recalling that $\alpha > 0$, by dividing each of these equations by α and making the change of variables $(\alpha, \beta, \delta, \varepsilon, s_0) \to (1, \frac{\beta}{\alpha}, \frac{\delta}{\alpha}, \frac{\varepsilon}{\alpha}, s_0)$, it is equivalent to set $\alpha = 1$ and consider the following constraints:

$$1 - 2s_0 + \delta - 2s_0\beta \geqslant 0; \quad (1) \qquad \beta + \delta > 0; \quad (2)$$

$$(a-1)\varepsilon + 1 + \beta \geqslant 0; \quad (3) \qquad \varepsilon - 1 - \beta \leqslant 0; \quad (4)$$

$$\varepsilon + 2\delta - (1-\beta) \geqslant 0; \quad (5) \qquad 2\varepsilon + (a-2)(1+\beta) + 2a\delta \leqslant 0. \quad (6)$$

The determination of the maximal s_0 subject to these constraints is a non-linear, non-convex optimisation problem falling within the scope of the Karush–Kuhn–Tucker theorem (originating from [77] from 1939 and [89] from 1951, but see also [145, Ch. 4] and [117, Ch. 11–12]) which is a generalisation of the method of Lagrange multipliers and can be stated as follows:

Theorem (Karush–Kuhn–Tucker) *For some fixed n, $m \in \mathbb{N}$, let f, $g_i \in C^1(\mathbb{R}^n)$ for each $i \in [1, m] \cap \mathbb{N}$. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$, define the Karush–Kuhn–Tucker–Lagrange function as:*

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x).$$

Suppose that $x_0 \in \mathbb{R}^n$ is such that the maximal value of $f(x)$ subject to the constraints $g_i(x) \geqslant 0$, $i \in [1, m] \cap \mathbb{N}$ is attained at $x = x_0$. Then necessarily there exists $\lambda \in \mathbb{R}^m$ such that

$$\nabla_x L(x, \lambda)\big|_{x=x_0} = 0$$

and for each $i \in [1, m] \cap \mathbb{N}$,

$$\lambda_i g_i(x_0) = 0 \quad \textit{and} \quad \lambda_i \geqslant 0.$$

Applying this theorem to the non-linear optimisation problem here yields the following system of equations to solve for λ_i ($i \in [1,6] \cap \mathbb{N}$, all non-negative) and the feasibly optimal choices of β, δ, ε and s_0:

$$\lambda_1(-2s_0) + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + (2-a)\lambda_6 = 0; \tag{i}$$

$$\lambda_1 + \lambda_2 + 2\lambda_5 - 2a\lambda_6 = 0; \tag{ii}$$

$$(a-1)\lambda_3 - \lambda_4 + \lambda_5 - 2\lambda_6 = 0; \tag{iii}$$

$$1 - 2(1+\beta)\lambda_1 = 0; \tag{iv}$$

$$\lambda_1(1 - 2s_0 + \delta - 2s_0\beta) = 0; \tag{v}$$

$$\lambda_2(\beta + \delta) = 0; \tag{vi}$$

$$\lambda_3((a-1)\varepsilon + 1 + \beta) = 0; \tag{vii}$$

$$\lambda_4(\beta + 1 - \varepsilon) = 0; \tag{viii}$$

$$\lambda_5(\varepsilon + 2\delta - (1-\beta)) = 0; \tag{ix}$$

$$\lambda_6((2-a)(1+\beta) - 2a\delta - 2\varepsilon) = 0. \tag{x}$$

The only solution to this system of equations with $\lambda_i \geqslant 0$ for each i and satisfying the original constraints is given by

$$\lambda = \left(\frac{1}{4(a-1)}, 0, \frac{1}{(a-1)a}\left(\frac{3}{8(a-1)} - \frac{1}{8}\right), 0, \frac{1}{8(a-1)}, \frac{1}{4a(a-1)}\right),$$
$$\beta = 2a - 3, \quad \delta = 3 - a, \quad \varepsilon = -2, \quad s_0 = \frac{3}{4(a-1)} - \frac{1}{4}.$$

Placing these values of β, δ, ε and s_0 into the above framework gives exactly the counterexample of Section 7.4.

It is remarked that counterexamples for the local problem (Theorem 7.1.1) can also be generated in this way, simply by adding the additional constraint

$$\boxed{\varepsilon \geqslant 0} \tag{7}$$

This modifies the Karush–Kuhn–Tucker conditions from the global problem by introducing an additional non-negative variable, λ_7, and an additional equation,

$$\lambda_7\varepsilon = 0, \tag{xi}$$

as well as replacing equation (iii) with the following:

$$(a-1)\lambda_3 - \lambda_4 + \lambda_5 - 2\lambda_6 + \lambda_7 = 0. \tag{iii'}$$

Again, there is only one solution set to this system of equations where all the λ_i are non-negative and the original constraints are satisfied:

$$\lambda = \left(\frac{1}{2a}, 0, 0, 0, \frac{1}{4a}, \frac{1}{2a^2}, \frac{1}{a^2} - \frac{1}{4a}\right),$$
$$\beta = a-1, \quad \delta = 1 - \frac{a}{2}, \quad \varepsilon = 0, \quad s_0 = \frac{1}{a} - \frac{1}{4}.$$

For $a \in [2,4)$, this constructs a counterexample similar to that of Vega from [143], obtaining the optimal value of s_0 of Vega's Theorem 7.1.1. The optimal regularity index is not recovered for $a \in (1,2)$, but the above scheme can also be adapted to form a counterexample similar to the one used by Vega in this case. Indeed, the function f can now be defined as $\widehat{f}(\xi) := e^{-i|\frac{1}{2}\xi|^A} v^\alpha g_v\left(v^\alpha \xi + \frac{1}{v^\beta}\right)$. This does not change the circumstances under which $\|f_v\|_{H^s(\mathbb{R})} \to 0$ as $v \to 0$. Assuming that $t > \frac{1}{2}$, $|R_a^t f_v(x)|$ can be written as

$$\left|\int_{-v^\delta}^{v^\delta} g_v(\eta)\phi\left(t\left(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}}\right)\right)\left|t\left(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}}\right)\right|^{-B} e^{i|(t-\frac{1}{2})(\frac{\eta}{v^\alpha} - \frac{1}{v^{\alpha+\beta}})|^A} e^{i(\frac{x\eta}{v^\alpha} - \frac{(t-\frac{1}{2})^A}{v^{A(\alpha+\beta)}})}\, d\eta\right|.$$

Setting $t - \frac{1}{2} = x^{\frac{1}{A}} v^{(\alpha+\beta)(1-\frac{1}{A})}$, the oscillatory terms can be controlled to be essentially constant as before. In particular, it is remarked that the translation in the choice of t effected by the introduction of the exponential term to the definition of $\widehat{f}$ allows control over the oscillatory terms in spite of an enforced constancy of t. Given that t must be local, this constancy is desirable for improving the control over the $|t_v(x)\xi|^{-B}$ term when $a<2$, since B is negative in this case.

Being specific, since $t\sim 1$,

$$\left|t\left(\frac{\eta}{v^{\alpha}}-\frac{1}{v^{\alpha+\beta}}\right)\right|^{-B}\sim v^{(\alpha+\beta)B},$$

so $|\widetilde{T}_a^* f_v(x)|\gtrsim v^{(\alpha+\beta)B+\delta}$ and hence $\|\widetilde{T}_a^* f_v\|_{L^2(\mathbb{R})}^2\gtrsim v^{\varepsilon}v^{2(\alpha+\beta)B+2\delta}$. To force $\|\widetilde{T}_a^* f_v\|_{L^2(\mathbb{R})}\gtrsim 1$ uniformly in v, constraint (6) becomes

$$\varepsilon+2(\alpha+\beta)B+2\delta\leqslant 0,$$

that is

$$\boxed{(a-1)\varepsilon+(1+\beta)(a-2)+2(a-1)\delta\leqslant 0}\tag{6$'$}$$

The resultant modified optimisation problem gives rise to the same Karush–Kuhn–Tucker equations as the previous local counterexample with the exception of equations (ii), (iii′) and (x) which are replaced with the following:

$$\lambda_1+\lambda_2+2\lambda_5+2(1-a)\lambda_6=0;\tag{ii$'$}$$

$$(a-1)\lambda_3-\lambda_4+\lambda_5+(1-a)\lambda_6+\lambda_7=0;\tag{iii$''$}$$

$$\lambda_6((1-a)\varepsilon+(1+\beta)(2-a)+2(1-a)\delta)=0.\tag{x$'$}$$

There is again only one solution which satisfies the necessary hypotheses, which is as follows:

$$\lambda=\left(\frac{1}{4(a-1)},0,0,0,\frac{1}{8(a-1)},\frac{1}{4(a-1)^2},\frac{1}{8(a-1)}\right),$$
$$\beta=2a-3,\quad \delta=2-a,\quad \varepsilon=0,\quad s_0=\frac{1}{2(a-1)}-\frac{1}{4}.$$

It can thus be seen that the optimal result from Theorem 7.1.1 for $a\in(1,2)$ is now recovered.

7.6 Further Results – The Complex Time Problem

Given the work of Chapter 6, it is natural to ask to what extent the work of the present chapter can be extended to the context of complex time.

For $A > 1$, $B \in (-\infty, \frac{1}{2})$, $\gamma > 0$ and $t \in (0,1)$, define the operator

$$Q^t_{A,B,\gamma} f(x) := \int_{\mathbb{R}} \widehat{f}(\xi)\phi(|t + it^\gamma|^{\frac{1}{A}}\xi)(|t + it^\gamma|^{\frac{1}{A}}|\xi|)^{-B} e^{i(t|\xi|^A + x\xi)} e^{-t^\gamma|\xi|^A}\, d\xi,$$

where ϕ is, as before, for example, a positive, even $C^\infty(\mathbb{R})$ function supported in $\mathbb{R}\backslash[-\frac{1}{4}, \frac{1}{4}]$ and equal to 1 in $\mathbb{R}\backslash[-\frac{1}{2}, \frac{1}{2}]$. Define the corresponding maximal operator,

$$Q^*_{A,B,\gamma} f := \sup_{t\in(0,1)} |Q^t_{A,B,\gamma} f|.$$

Also, for $a > 1$, define $Q^t_{a,\gamma} := Q^t_{\frac{a}{a-1}, \frac{a-2}{2(a-1)},\gamma}$ and $Q^*_{a,\gamma} := Q^*_{\frac{a}{a-1}, \frac{a-2}{2(a-1)},\gamma}$. These operators correspond to the earlier operators $\widetilde{T}^t_a$ and $\widetilde{T}^*_a$ with time $t + it^\gamma$.

The remainder of this section will consist of proving and discussing boundedness results for $Q^*_{a,\gamma}$, the first of which is the following theorem:

Theorem 7.6.1 *For each $a > 1$ and $\gamma > 0$, denote by $\widetilde{s}_a(\gamma)$ the infimum of the non-negative s such that*

$$\|Q^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for all $f \in H^s(\mathbb{R})$. The following results hold:

(i) For $a \in (1,2]$, $\widetilde{s}_a(\gamma) \in [\frac{1}{a-1}(1 - \frac{1}{\gamma})^+(1 - \frac{a}{4}), \frac{1}{a-1}(1 - \frac{a}{4}(1 + \frac{1}{\max(1,\gamma)}))]$;

(ii) For $a \in [2,4)$, $\widetilde{s}_a(\gamma) = \frac{1}{a-1}(1 - \frac{1}{\gamma})^+(1 - \frac{a}{4})$;

(iii) For $a \geqslant 4$, $\widetilde{s}_a(\gamma) = 0$.

Proof The positive parts of this theorem can be proved using the same method as in the proof of Theorem 7.1.2, drawing on Theorem 6.1.2 on the boundedness of the Schrödinger-type maximal operators with complex-valued time instead of the 1994 result of Sjölin used there. To begin with, note that Theorem 7.3.1 of Vega can be adapted without difficulty to apply to the operators here and the result for $a > 4$ follows as a consequence. Indeed, Theorem 7.3.1 proved that for $A > 1$, $B > \frac{A}{4}$ and $f \in \mathscr{S}(\mathbb{R})$,

$$\| \sup_{t\in\mathbb{R}^+} |\widetilde{T}^t_{A,B} f| \|_{L^2(\mathbb{R})} \lesssim_{A,B} \|f\|_{L^2(\mathbb{R})}.$$

Regardless of the choice of $\gamma > 0$, Lemma 7.3.2 may be adapted (following the procedures of the proof of Lemma 6.2.1 in Section 6.2) to establish the same result for the oscillatory integrals generated by these operators with complex-valued time. It can be concluded that for $A > 1$, $B > \frac{A}{4}$, $\gamma > 0$ and $f \in \mathscr{S}(\mathbb{R})$,

$$\| \sup_{t\in\mathbb{R}^+} |Q^t_{A,B,\gamma} f| \|_{L^2(\mathbb{R})} \lesssim_{A,B} \|f\|_{L^2(\mathbb{R})}.$$

Considering that for $a > 1$, the operator $Q^t_{a,\gamma}$ corresponds to $Q^t_{A,B,\gamma}$ for $A = \frac{a}{a-1}$ and $B = \frac{a-2}{2(a-1)}$ and that these values of A and B satisfy the inequality $B > \frac{A}{4}$ precisely when $a > 4$, the result follows in this case.

For $a \in (1,2]$, note as before that when $A = \frac{a}{a-1}$ and $B = \frac{a-2}{2(a-1)}$, B is non-positive, so for each $x \in \mathbb{R}$,

$$Q^*_{a,\gamma} f(x) \leqslant \sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \widehat{f_B}(\xi) \phi(|t + it^\gamma|^{\frac{1}{A}} \xi) e^{i(t|\xi|^A + x\xi)} e^{-t^\gamma |\xi|^A} \, d\xi \right|,$$

where $\widehat{f_B} = |\cdot|^{-B} \widehat{f}$. As before, the proof of Theorem 6.1.2 is equally applicable to the above operator with its added smooth cut-off function, ϕ, so

$$\|Q^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f_B\|_{H^s(\mathbb{R})}$$

for any $s > \frac{1}{4}A(1-\frac{1}{\gamma})^+$. Since it can again be shown that $\|f_B\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^{s-B}(\mathbb{R})}$, it follows that

$$\|Q^*_{a,\gamma}f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for any $s > \frac{1}{4}A(1-\frac{1}{\gamma})^+ - B = \frac{1}{a-1}(1-\frac{a}{4}(1+\frac{1}{\max(1,\gamma)}))$. The upper bound for $\widetilde{s}_a(\gamma)$ is thus established for $a \in (1,2]$.

An upper bound for $a \in [2,4)$ again follows using Stein's interpolation theorem. Indeed, fixing $A > 1$ and considering B to be an interpolation parameter between 0 and $\frac{A}{4}$, interpolating between the adaptation of Theorem 7.3.1 used to prove the $a > 4$ result and the bound from Theorem 6.1.2 used to prove the positive part of the $a \in (1,2]$ result, it follows that for $\theta \in (0,1)$,

$$\|Q^*_{A,\theta\frac{A}{4},\gamma}f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for $s > \frac{1}{4}A(1-\frac{1}{\gamma})^+(1-\theta)$. To conclude the desired result for $Q^*_{a,\gamma}$, set $A = \frac{a}{a-1}$ and $\theta = 2 - \frac{4}{a}$ so that $\theta\frac{A}{4} = \frac{a-2}{2(a-1)}$. The corresponding lower bound for s is

$$\frac{a}{4(a-1)}\left(1-\frac{1}{\gamma}\right)^+\left(\frac{4}{a}-1\right) = \frac{1}{a-1}\left(1-\frac{1}{\gamma}\right)^+\left(1-\frac{a}{4}\right),$$

as required.

It remains to prove the negative parts of the theorem for $a \in (1,4)$. To do this, it suffices to consider only $\gamma > 1$, since the stated lower bounds on $\widetilde{s}_a(\gamma)$ become 0 for $\gamma \in (0,1]$. The first scheme from Section 7.5 for generating counterexamples can be applied to these operators*, simply adding the constraint that the term $e^{-t^\gamma|\xi|^A}$ is bounded below. Within the context of that scheme, this term corresponds to

$$\exp(-O(v^{\varepsilon\gamma+(\alpha+\beta)(\gamma(A-1)-A)}))$$

*A transformation, $t^A \mapsto t$, is required, which has no impact on the validity of the methods.

and so a suitable additional constraint is

$$\boxed{\varepsilon\gamma + (\alpha+\beta)(\gamma(A-1)-A) \geqslant 0} \tag{7}$$

As before, α can be set to 1, so the generation of counterexamples corresponds to maximising $s_0 \geqslant 0$ subject to the following constraints:

$$1 - 2s_0 + \delta - 2s_0\beta \geqslant 0; \quad (1) \qquad \beta + \delta > 0; \quad (2)$$

$$(a-1)\varepsilon + 1 + \beta \geqslant 0; \quad (3) \qquad \varepsilon - 1 - \beta \leqslant 0; \quad (4)$$

$$\varepsilon + 2\delta - (1-\beta) \geqslant 0; \quad (5) \qquad 2\varepsilon + (a-2)(1+\beta) + 2a\delta \leqslant 0; \quad (6)$$

$$\varepsilon\gamma(a-1) + (1+\beta)(\gamma - a) \geqslant 0. \quad (7)$$

The equations generated for this optimisation problem from the Karush–Kuhn–Tucker Theorem are as follows:

$$\lambda_1(-2s_0) + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + (2-a)\lambda_6 + (\gamma - a)\lambda_7 = 0; \tag{i}$$

$$\lambda_1 + \lambda_2 + 2\lambda_5 - 2a\lambda_6 = 0; \tag{ii}$$

$$(a-1)\lambda_3 - \lambda_4 + \lambda_5 - 2\lambda_6 + \gamma(a-1)\lambda_7 = 0; \tag{iii}$$

$$1 - 2(1+\beta)\lambda_1 = 0; \tag{iv}$$

$$\lambda_1(1 - 2s_0 + \delta - 2s_0\beta) = 0; \tag{v}$$

$$\lambda_2(\beta + \delta) = 0; \tag{vi}$$

$$\lambda_3((a-1)\varepsilon + 1 + \beta) = 0; \tag{vii}$$

$$\lambda_4(\beta + 1 - \varepsilon) = 0; \tag{viii}$$

$$\lambda_5(\varepsilon + 2\delta - (1-\beta)) = 0; \tag{ix}$$

$$\lambda_6((2-a)(1+\beta) - 2a\delta - 2\varepsilon) = 0; \tag{x}$$

$$\lambda_7(\varepsilon\gamma(a-1) + (1+\beta)(\gamma - a)) = 0. \tag{xi}$$

This system of equations has the following unique solution such that the λ_i are all non-negative and the constraints are satisfied:

$$\lambda = \left(\frac{1}{4}\frac{\gamma+a-2}{\gamma(a-1)}, 0, 0, 0, \frac{1}{8}\frac{\gamma+a-2}{\gamma(a-1)}, \frac{1}{4}\frac{\gamma+a-2}{\gamma a(a-1)}, \frac{1}{8}\frac{4\gamma+6a-8-\gamma a-a^2}{(a^2-2a+1)\gamma^2 a}\right),$$
$$\beta = \frac{2\gamma a - 3\gamma - a + 2}{\gamma+a-2}, \quad \delta = \frac{3\gamma-2-\gamma a}{\gamma+a-2}, \quad \varepsilon = \frac{2a-2\gamma}{\gamma+a-2}, \quad s_0 = \frac{1}{a-1}\left(1-\frac{1}{\gamma}\right)\left(1-\frac{a}{4}\right).$$

It follows that counterexamples for boundedness exist for all s smaller than this value of s_0 and so the proof of the theorem is complete. $\square$

As in Chapter 6, local bounds for $Q^*_{a,\gamma}$ can be deduced in a straightforward manner using the global bounds and their proof. The following holds:

Theorem 7.6.2 *For each $a > 1$ and $\gamma > 0$, denote by $\widetilde{s}_a^{\mathrm{loc}}(\gamma)$ the infimum of the non-negative s such that*

$$\|Q^*_{a,\gamma} f\|_{L^2([-1,1])} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for all $f \in H^s(\mathbb{R})$. The following results hold:

(i) For $a \in (1,2]$,

$$\widetilde{s}_a^{\mathrm{loc}}(\gamma) \in \left[\min\left(\tfrac{1}{a-1}(1-\tfrac{1}{\gamma})^+(1-\tfrac{a}{4}), \tfrac{4-a}{4a}\right), \min\left(\tfrac{1}{a-1}(1-\tfrac{a}{4}(1+\tfrac{1}{\max(1,\gamma)})), \tfrac{3-a}{4(a-1)}\right)\right];$$

(ii) For $a \in [2,4)$, $\widetilde{s}_a^{\mathrm{loc}}(\gamma) = \min\left(\tfrac{1}{a-1}(1-\tfrac{1}{\gamma})^+(1-\tfrac{a}{4}), \tfrac{4-a}{4a}\right)$;

(iii) For $a \geqslant 4$, $\widetilde{s}_a^{\mathrm{loc}}(\gamma) = 0$.

Proof Since $\|\cdot\|_{L^2([-1,1])} \leqslant \|\cdot\|_{L^2(\mathbb{R})}$, it is certainly the case that $\widetilde{s}_a^{\mathrm{loc}}(\gamma) \leqslant \widetilde{s}_a(\gamma)$ for all $a > 1$ and $\gamma > 0$. It thus remains to prove the lower bounds on $\widetilde{s}_a^{\mathrm{loc}}(\gamma)$ for $a \in (1,4)$ and $\gamma > 1$ and the upper bounds for $a \in (1,4)$ and $\gamma > a$. Now, the counterexamples for the global problem are also valid here when $\varepsilon := \frac{2a-2\gamma}{\gamma+a-2}$ is non-negative and for $a, \gamma > 1$, this happens precisely when $\gamma \leqslant a$, so the results of Theorem 7.6.1 transfer directly for all $a > 1$ and $\gamma \in (0, a]$.*

*Note that the point $\gamma = a$ corresponds to $\gamma = \frac{A}{A-1}$ for $A = \frac{a}{a-1}$; that there is a change in behaviour of $\widetilde{s}_a^{\mathrm{loc}}(\gamma)$ at this point is natural in light of the change in behaviour of $s_a^{\mathrm{loc}}(\gamma)$ at $\gamma = \frac{a}{a-1}$ in Theorem 6.4.2.

As in the proof of Theorem 6.4.2, counterexamples generated for any particular value of γ remain valid for higher values of γ, so it is certainly the case that $\widetilde{s}_a^{\text{loc}}(\gamma) \geqslant \widetilde{s}_a^{\text{loc}}(a)$ for all $\gamma \geqslant a$. It thus only remains to prove positive boundedness results for $a \in (1,4)$, $\gamma > a$.

For $a \in (1,2]$, proceeding exactly as in the proof of Theorem 7.6.1, but using the local complex time results of Theorem 6.4.2 instead of the global results of Theorem 6.1.2, it can be seen that for any $\gamma > a$, boundedness occurs for $s > \frac{1}{4} - B$, where $B = \frac{a-2}{2(a-1)}$, in other words, for $s > \frac{3-a}{4(a-1)}$. The result for $a \in (2,4)$ also follows by repeating the arguments from the proof of Theorem 7.6.1 but using Theorem 6.4.2; in this case boundedness is established for $s > \frac{1}{4}(1-\theta)$ for $\theta = 2 - \frac{4}{a}$, that is to say that boundedness occurs for $s > \frac{4-a}{4a}$, as required. □

It is clear that there is further work to do to complete the statements of Theorems 7.6.1 and 7.6.2 in the case of $a \in (1,2)$. In both cases, there seem to be two main difficulties in providing a precise value for the threshold Sobolev index.

The first of these difficulties is that the upper bounds on $\widetilde{s}_a(\gamma)$ and $\widetilde{s}_a^{\text{loc}}(\gamma)$ are increased for $\gamma < 1$ by the presence of the term "$\max(1,\gamma)$" in place of simply "γ". The latter term seems like a very reasonable conjecture in light of the behaviour of $\widetilde{s}_a(\gamma)$ and $\widetilde{s}_a^{\text{loc}}(\gamma)$ for $a \in [2,\infty)$ as well as the fact that the "$\max(1,\gamma)$" term was only required because the complex time results of Chapter 6 drawn upon to prove Theorems 7.6.1 and 7.6.2 have not considered the possibility of results for Sobolev spaces of negative index and thus bounds from L^2 to L^2 are the best available for $\gamma < 1$.

The second difficulty is that for all $a \in (1,4)$, the proposed scheme for generating counterexamples only disproves boundedness below the Sobolev index above which boundedness has been shown to occur for $a \in [2,4)$. This index is smaller than the proved boundedness threshold for $a \in (1,2)$. Identical behaviour was seen in Section 7.5 when generating counterexamples that confirmed Vega's local result, Theorem 7.1.1. There, the solution was to multiply the functions $\widehat{f_v}$ by a fixed modulating function, $e^{-i|\frac{1}{2}\xi|^A}$; this allows t to remain bounded below by a fixed positive constant as v tends to 0 whilst still allowing the oscillatory term of the integrand to be controlled to be essentially constant in the same way as before.

Unfortunately, the same argument does not appear to work here, since when t is essentially constant, the need to bound $(|t+it^{\gamma}|^{\frac{1}{A}}|\xi|)^{-B}$ below and the need to bound $e^{-t^{\gamma}|\xi|^{A}}$ below are conflicted. Nonetheless, it is curious to note that the equations produced from the Karush–Kuhn–Tucker theorem for an argument of this form do produce $s_0 = \frac{1}{a-1}(1-\frac{a}{4}(1+\frac{1}{\gamma}))$ as a solution. Indeed, for $\gamma > 1$, following the construction of a counterexample for Vega's theorem at the end of Section 7.5 as a guide, bounding $e^{-t^{\gamma}|\xi|^{A}}$ below in this case amounts to controlling

$$\exp(-(O(v^{-A(\alpha+\beta)})+O(v^{\varepsilon\gamma+(\alpha+\beta)(\gamma(A-1)-A)}))).$$

This generates two constraints,

$$\boxed{A(\alpha+\beta)\leqslant 0}$$

and

$$\boxed{\varepsilon\gamma+(\alpha+\beta)(\gamma(A-1)-A)=0}$$

Choosing $\alpha = 1$ as usual, a suitable final optimisation problem is to maximise s_0 subject to the following constraints:

$$1-2s_0+\delta-2s_0\beta \geqslant 0; \quad (1) \qquad \beta+\delta > 0; \quad (2)$$

$$(a-1)\varepsilon+1+\beta \geqslant 0; \quad (3) \qquad \varepsilon-1-\beta \leqslant 0; \quad (4)$$

$$\varepsilon+2\delta-(1-\beta) \geqslant 0; \quad (5) \qquad (a-1)\varepsilon+(a-2)(1+\beta)+2(a-1)\delta \leqslant 0; \quad (6)$$

$$\beta+1 \leqslant 0; \quad (7) \qquad (a-1)\varepsilon\gamma+(1+\beta)(\gamma-a) \geqslant 0. \quad (8)$$

The corresponding equations generated from the Karush–Kuhn–Tucker theorem have the following solution:

$$\lambda = \left(\frac{1}{4(a-1)}, 0, 0, 0, \frac{1}{8(a-1)}, \frac{1}{4(a-1)^2}, 0, \frac{1}{8\gamma(a-1)^2}\right),$$

$$\beta = 2a - 3, \quad \delta = 3 - a\left(1 + \frac{1}{\gamma}\right), \quad \varepsilon = \frac{2a}{\gamma} - 2, \quad s_0 = \frac{1}{a-1}\left(1 - \frac{a}{4}\left(1 + \frac{1}{\gamma}\right)\right).$$

Nonetheless, this solution can be seen to fail to satisfy constraint (7), which is a direct manifestation of the conflict described.

It is natural to attempt to adapt this scheme further by introducing an additional parameter, θ, and replacing the fixed modulation function that the functions $\widehat{f_v}$ were multiplied by with $e^{-iv^\theta|\xi|^A}$; by doing this, the derived choice of t still tends to 0, removing the conflict that was present with the fixed modulation, yet some greater control over the order to which it vanishes is offered. One might also attempt to weaken some of the constraints that were taken directly from Section 7.5 by considering more carefully where the term "$|t + it^\gamma|$" might provide better bounds than simply "t". However, despite carefully adapting the scheme of Section 7.5 to incorporate these ideas, the author has not yet been able to produce counterexamples that improve either Theorem 7.6.1 or Theorem 7.6.2.

Concluding Remarks for Part II

Naturally, the most apparent open problem arising from Part II of this thesis is "completing" Theorems 7.6.1 and 7.6.2. In light of the discussion at the end of Section 7.6, the most natural conjecture would seem to be the following:

Conjecture *For each $a > 1$ and $\gamma > 0$, denote by $\widetilde{s}_a(\gamma)$ the infimum of the non-negative s such that*

$$\|Q^*_{a,\gamma} f\|_{L^2(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for all $f \in H^s(\mathbb{R})$ and by $\widetilde{s}^{\,\mathrm{loc}}_a(\gamma)$, the infimum of the non-negative s such that

$$\|Q^*_{a,\gamma} f\|_{L^2([-1,1])} \lesssim \|f\|_{H^s(\mathbb{R})}$$

for all $f \in H^s(\mathbb{R})$. The following results hold:

(i) For $a \in (1,2]$,
- $\widetilde{s}_a(\gamma) = \frac{1}{a-1}(1 - \frac{a}{4}(1+\frac{1}{\gamma}))^+$;
- $\widetilde{s}^{\,\mathrm{loc}}_a(\gamma) = \min\left(\frac{1}{a-1}(1 - \frac{a}{4}(1+\frac{1}{\gamma}))^+, \frac{3-a}{4(a-1)}\right)$;

(ii) For $a \in [2,4)$,
- $\widetilde{s}_a(\gamma) = \frac{1}{a-1}(1 - \frac{1}{\gamma})^+(1-\frac{a}{4})$;
- $\widetilde{s}^{\,\mathrm{loc}}_a(\gamma) = \min\left(\frac{1}{a-1}(1 - \frac{1}{\gamma})^+(1-\frac{a}{4}), \frac{4-a}{4a}\right)$

(iii) For $a \geqslant 4$, $\widetilde{s}_a(\gamma) = \widetilde{s}^{\,\mathrm{loc}}_a(\gamma) = 0$.

To prove this conjecture, it would suffice to address the two difficulties discussed in Section 7.6, improving for $a \in (1,2)$ the proofs of boundedness for $\gamma \in (0,1)$ and the counterexamples for $\gamma \in (0,\infty)$; the remainder of the conjecture is already contained in Theorems 7.6.1 and 7.6.2. The extent to which the methods of Section 7.6 need improving is at present unclear.

Another point worthy of further study is the question of whether the results and methods on the "maximal multiplier operators", $\widetilde{T}^*_a$, of Chapter 7 can be used to prove the corresponding results for the "maximal kernel operators", T^*_a. As was discussed in Section 7.2, these operators are intimately related via the asymptotic expansion of Miyachi[105], but the established error bound is not sufficient to show an equivalence of boundedness in a direct way. Whilst an improved error bound, if possible, would naturally provide the most direct route here, it is possible that properties of the operators T_a^* could be exploited to allow the results of Chapter 7 to be transferred to this context in a less routine way. Alternatively, if additional information about the first part of the error term of Proposition 7.2.1, $e_a(x)e^{ic_a|x|^a}$, could be established, it is possible that the maximal operator corresponding to this term could be bounded directly; as remarked in Section 7.2, the maximal operator corresponding to the remaining error term of Proposition 7.2.1 can be pointwise bounded by the Hardy–Littlewood maximal operator, so bounds for the operators T_a^* would follow from Theorems 7.1.1 and 7.1.2.

A natural generalisation of the work of Chapter 6 is to consider the problem for $a = 1$. For real-valued time, t, the operator S_1^t essentially corresponds to the solution operator for the wave equation. Boundedness properties of the corresponding maximal operators were included in the abstract work of Cowling mentioned in Section 5.1[50] and have more recently been considered directly in a 2008 paper of Rogers and Villarroya[121]. Whilst the methods used in this thesis to bound the complex time operators $P_{a,\gamma}^*$ are highly dependent upon the requirement that $a > 1$, it is clear that these operators are of quite a similar nature for $a = 1$ and $a > 1$, with a rapidly decaying factor introduced to the real-valued time operator multiplier by the imaginary part of the temporal variable in both cases. It is thus reasonable to expect that methods used to study the boundedness of the operators S_1^* from [121], for example, might be adapted and combined with ideas from Chapter 6 to prove and disprove boundedness results for $P_{a,\gamma}^*$.

In his previously cited paper from 1994[130], Sjölin showed for $a > 1$ that the globally maximal operator $\sup_{t\in\mathbb{R}}|S_a^t\cdot|$ is not bounded from $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$ for any $s \geqslant 0$. Nonetheless, it is clear that the globally maximal operator corresponding to the solution operator for the heat equation, $\sup_{t\in\mathbb{R}}|S_a^{it}\cdot|$, is bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ (given that it is controlled by the Hardy–Littlewood maximal operator). In light of this fact, it is reasonable to ask for $\gamma > 0$ what

$H^s(\mathbb{R}) \to L^2(\mathbb{R})$ bounds the globally maximal complex time operator $\sup_{t\in\mathbb{R}} |S^{t+it^\gamma} \cdot|$ satisfies. It is natural to conjecture that such bounds can be established and that the infimum of the values of $s > 0$ for which a bound from $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$ holds will tend to 0 as γ tends to 0 and to infinity as γ tends to infinity.*

It is remarked in Section 6.1 that Lemma 6.1.4 suggests that the operators $P^*_{a,\gamma}$ are natural operators to consider with relation to the problem of pointwise convergence at the origin of Schrödinger operators with complex time as they encapsulate the convergence properties of any operator of the form $S_a^{t+ih(t)}$ when $h(t)$ is of polynomial type near $t = 0$. Nonetheless, it would also be reasonable to study boundedness of operators of the form $\sup_{t\in(0,1)} |S_a^{g_\theta(t)+ih_\theta(t)} \cdot|$ for more general functions $g_\theta : [0,1] \to [0,1]$ and $h_\theta : [0,1] \to [0,1]$ such that $g_\theta(0) = h_\theta(0) = 0$. The corresponding maximal operator is the following:

$$\sup_{t\in(0,1)} \left| \int_{\mathbb{R}} \widehat{f}(\xi) e^{ig_\theta(t)|\xi|^a} e^{-h_\theta(t)|\xi|^a} e^{ix\xi}\, d\xi \right|.$$

Much of the existing analysis of Chapter 6 carries through to this more general situation. The integral of interest in Lemma 6.2.1 becomes the following:

$$\left| \int_{\mathbb{R}} e^{i((g_\theta(t_1)-g_\theta(t_2))|\xi|^a - x\xi)} (1+\xi^2)^{-\frac{\alpha}{2}} e^{-(h_\theta(t_1)+h_\theta(t_2))|\xi|^a} \mu\left(\frac{\xi}{N}\right) d\xi \right|.$$

The goal then remains to find values of α for each θ such that the above can be bounded uniformly by a function in $L^1(\mathbb{R})$. By adapting the definitions of "t" and "ε" from the proof of Lemma 6.2.1 to suit this integral (either in a general setting or for some specific choices of g_θ and h_θ), the majority of the proof remains applicable. Perhaps unsurprisingly, the bulk of the required additional work is in establishing bounds for the "J_2" terms in Sections 6.2.1 and 6.2.2, although it is also necessary to consider the possibility of sign changes in the newly defined "t".

*The author would like to thank Dr. David Rule for suggesting this problem.

One particularly natural candidate for a reformulation of the work of Chapter 6 of the above form is where $g_\theta(t) := t\cos(\theta)$ and $h_\theta(t) := t\sin(\theta)$ for $\theta \in [0, \frac{\pi}{2}]$.* Here, the paths $t + it^\gamma$ of Chapter 6 have been replaced with rays in the complex plane of various slopes, $t(\cos(\theta) + i\sin(\theta))$; this situation has the advantage that the above maximal operator transforms exactly into the Schrödinger maximal operator for $\theta = 0$ and into the maximal operator for the heat equation for $\theta = \frac{\pi}{2}$. Nonetheless, given that these slopes are of positive constant gradient for each $\theta \in (0, \frac{\pi}{2})$ and given the earlier comment about the results for the paths $t + it^\gamma$ encapsulating the situation of convergence of polynomial type at the origin, one might expect the boundedness results for these rays in the complex plane to correspond in all cases to the case of $\gamma = 1$ of Theorem 6.1.2, that is $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ boundedness. In other words, it seems reasonable to hypothesise that the established bounds resemble those for the maximal operator for the heat equation throughout $\theta \in (0, \frac{\pi}{2})$ and do not exhibit any kind of transition to the situation for the Schrödinger maximal operator as θ approaches 0.

This conjecture seems to be supported by the above reasoning for general g_θ and h_θ. Indeed, in this context, the terms "t" and "ε" in the proof of Lemma 6.2.1 are defined as

$$\begin{aligned} t &:= (t_1 - t_2)\cos(\theta); \\ \varepsilon &:= (t_1 + t_2)\sin(\theta). \end{aligned}$$

Since $\cos(\theta)$ does not change sign in $[0, \frac{\pi}{2}]$, the proof of Lemma 6.2.1 in the case of $\gamma = 1$ can be repeated almost identically here. In the bound on J_2 in Section 6.2.1, the inequality $\varepsilon \gtrsim_\gamma t$ (which follows from the inequality $t_1^\gamma + t_2^\gamma \gtrsim_\gamma t^\gamma$, which is simply $t_1 + t_2 \geqslant t_1 - t_2$ in the case of $\gamma = 1$) is replaced with the inequality $\varepsilon \geqslant \tan(\theta)t$. Whilst this bound degenerates for $\theta \in \{0, \frac{\pi}{2}\}$, the bound on J_2 in Section 6.2.1 in the case of $\gamma = 1$ seems to follow through with this adaptation for all $\theta \in (0, \frac{\pi}{2})$. The same remark applies to the argument in Section 6.2.2.†

*The author would like to thank Professor James Wright for suggesting this problem.

†As an alternative to following this line of reasoning, Lemma 6.1.4 could feasibly be applied directly to this situation. Indeed, by rescaling in t, the paths $t\cos(\theta) + it\sin(\theta)$ could be rewritten as $t + it\tan(\theta)$. With some care to resolve the corresponding adjustment to the domain of t, the same result would follow.

Whilst all the results in this part of this thesis have been about boundedness of operators from H^s to L^2, there has been extensive study of H^s to L^p bounds for the Schrödinger maximal operator for general $p \in [1, \infty]$; the reader is referred to [120] for a summary of such results. Another natural generalisation of the boundedness results for the operators herein would thus be to consider what H^s to L^p bounds can be proved.

Finally, it is remarked that the ideas of Section 7.5 can be considered in a much wider context than that of the theorems proved here. There is a general philosophy arising from Section 7.5, which applies to generation of counterexamples for any parameter-dependent operator bounds, namely that known counterexamples might be found to be susceptible to generalisation by introducing arbitrary parameters in appropriate places and generating a corresponding optimisation problem. Counterexamples for bounds that do not depend on a parameter can also be generated in this way; in this case the problem reduces to simply finding a solution for a system of inequalities without any need for optimisation.

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Note: To ensure that references given are authoritative, except in the case of the names of authors of publications in Russian, which have been transliterated to Roman script for reasons of practicality, all surnames and article titles herein are presented exactly as in the original publication (with translations provided, where appropriate) without regard for consistency of presentation.

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